UNIVERSITY of TASMANIA

Remediation of Errors with Mathematical Algorithms

by

Alison J. Manson, BHM, MEd

Thesis Submitted in Fulfilment

of the Requirements for the Degree of

Doctor of Philosophy

Faculty of Education

University of Tasmania

2019

Declaration of Originality

This thesis contains no material which has been accepted for a degree or diploma by the University or any other institution, except by way of background information and duly acknowledged in the thesis, and to the best of my knowledge and belief it contains no material previously published or written by another person except where due acknowledgement is made in the text or the thesis, nor does the thesis contain any material that infringes copyright.

Signature:

Date: 21/08/2018

Authority of Access

Signature:

Date: 21/08/2018

Statement of Ethical Conduct

The research associated with this thesis abides by the international and Australian codes on human and animal experimentation, the guidelines by the Australian Government's Office of the Gene Technology Regulator, and the rulings of the Safety, Ethics and Institutional Biosafety Committees of the University.

Signature:

Date: 21/08/2018

Abstract

This thesis is concerned with students' fraction understanding and the effects of remedial instruction on understanding, computational skills, and self-efficacy. The study makes the extent of fraction misconceptions among secondary students visible, supporting the literature that achieving a depth of understanding in fractions is both complex and difficult. Students do not construct meaning in isolation; rather, they try to make sense of new ideas based on what they already know. As students grapple with the conceptual development of fractions from natural numbers, they must also make sense of the complex manipulation of procedures. Students often do not remember which procedural operation to use when doing fraction computation and, coupled with a lack of deep understanding, often do not experience success with fractions. As a result, students become despondent about their ability and achievement in the topic, leading to low self-efficacy.

Past research has revealed that inappropriate application of prior knowledge causes an interference effect, which can result in erroneous procedures. The interference effect, known as *proactive inhibition* (Underwood, 1957), impacts on learning and memory by conflicting associations of prior learning. This thesis explores the effect proactive inhibition has on the learning of fractions and the effect is used to explain how inappropriate prior knowledge results in the misapplication of fraction procedures. Our knowledge of typical errors in fraction computation enables us to identify students who have difficulties performing standard fraction operations.

This large-scale study was conducted in an authentic school setting with students from years seven, eight, and nine (n=361) participating. Drawing on the literature about fraction misconceptions, an instrument was developed to expose fraction errors and to allow the diagnosis of repeat error patterns. The research confirmed the commonality of certain fraction misconceptions and highlights a lack of conceptual understanding.

Students identified as having misconceptions (n=83) were invited to participate in one of two remediation programs. One program, the Old Way / New Way technique (Lyndon, 1989), designed to counteract the effect of proactive inhibition, brings the learner's "old way" to a conscious level and exchanges it for a "new way" by means of discrimination learning. The effectiveness of this method was examined, in comparison to a traditional re-teaching technique. The programs ran concurrently for five weeks, with two sessions per week. The effectiveness of the two intervention strategies was determined through the analysis of pre-, post- and delayed retention test results. Pre- and post- self-efficacy was also examined, to determine the effect the intervention programs had on students' confidence in their ability to perform fraction algorithms.

The Old Way / New Way (O/N) intervention students gained significantly better pre-post results; however, the effectiveness of the O/N method was not maintained in the delayed retention test. Although the students verbally reported having more confidence after the intervention, this was not fully reflected in the self-efficacy scale. The three psychological domains of functioning were examined in the self-efficacy scale. Questions related to the affective domain examined students' internal belief system;

awareness of their own mathematical knowledge was examined in the cognitive domain questions; and questions about the conative domain looked at students' striving and level of focused attention in mathematics. Self-efficacy improved significantly in the conative domain pre-post for both intervention groups; however, there was no change in the cognitive domain and reliability was not able to be achieved for the affective domain scores.

Results of this research highlighted the interference effect of prior knowledge, with students not remembering procedures for specific operations and lacking the conceptual understanding to support their responses. Although students want to apply an operational procedure, they often do not have good recall of appropriate procedures, due to the interference effect. Remediation was shown to have an impact on the learning of fractions and to improve self-efficacy, with the Old Way / New Way method yielding significant short-term gains, but such remediation may need to be conducted regularly for long-term impact. This thesis discusses what can be achieved with fraction remediation and discusses whether techniques such as the Old Way / New Way intervention might be able to be used more widely in mathematics.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

1.1 Background

Past learning often affects future learning and memory for previously learned information by exerting either facilitation or interference effects (Darby & Sloutsky, 2015). Prior knowledge is one of the most important prerequisites for learning (Ausubel, 1968) and, as mathematics learning is cumulative and learning must be meaningful, new learning must be attached to what is already known. Prior knowledge for mathematics learning includes previous concepts of the learning trajectory, vocabulary, informal experiences, generalisations, and conceptual frameworks. It is this knowledge that assists learning by providing connection for new learning. A student's existing schema will determine what new information is learnt, but this prior understanding can impede learning if misconceptions and erroneous knowledge are part of those schema. What is taught is not always learned. What is learned is what is understood, but what is understood may be erroneous. If a student is unable to "assimilate" and "accommodate" new concepts correctly (Piaget, 1972), this creates a gap or misconstruction in the learning of a concept, which in turn, leads to mathematical misconceptions.

Although making errors is a significant part of the learning process, there are instances where students connect new information with pre-conceived knowledge where either the preconceptions are inappropriate or the new connections are not correctly constructed. Prior knowledge is not always correct knowledge; therefore, it is important that we assess the preconceptions of students and deal with them when found to be incorrect. Unless interventions are conducted by teachers, these errors will persist.

Deficiencies in prior relevant knowledge hinders many students' acquisition of fraction arithmetic, as a specific example, and a significant body of research has focused attention on the development of rational number knowledge from students' understanding of whole number (see, e.g., Chinnappan & Lawson, 2002). The multifaceted nature of the rational number domain makes learning difficult, so it is little wonder that students have difficulty understanding fractions and often scaffold their understanding on false pretexts.

Findings from various studies (e.g. Behr, Harel, Post, & Lesh, 1992; Bezuk & Bieck, 1993; Baturo & Cooper, 1995; Owen & Super, 1993; Chinnappan & Lawson, 2002) suggest that there is a need to investigate how teachers can enhance the fraction understanding of students and eradicate misconceptions. The research presented in this thesis responds to the question of how an intervention program, which focuses on the interference effect of prior learning, might contribute to assisting students with fraction misconceptions. The effectiveness of the intervention was examined in comparison to a traditional intervention program, and the impact that the intervention had on mathematical self-efficacy was investigated. The context of the present study is elaborated on in the first section of this chapter. The statement of the problem follows the context of the study and the aim and scope of the study are then discussed. The potential impact of the research is described in the section outlining the significance of the study and the chapter concludes by detailing the structure of the thesis.

1.2 Context

Fractions are known to be problematic (Brousseau, Brousseau & Warfield, 2004; Kieren, 1976; Lamon, 2001; Newstead & Murray, 1998). Despite there being a move to teaching and learning that involves a more conceptual base of fraction understanding of the numbers themselves before operations are introduced in the curriculum, students worldwide are still grappling with the concept of a fraction and which procedures to use to solve fraction problems. Fractions have been described as a predictor of success in overall mathematics achievement and provide students with important prerequisite conceptual foundations for the growth and understanding of other number types and algebraic operations in the later years of mathematics. For this reason, it is important that we assist students with their understanding of fraction concepts even after the minimum requirements of the curriculum achievement standards have been taught, and that we continue to identify and remedy misconceptions of fraction understanding.

In Australia, the concept of a fraction is taught at an early age, with students in Year 1 needing to recognise and describe one-half as one of two equal parts of a whole (ACARA, 2017). In Year 5 students investigate strategies to solve problems involving addition and subtraction of fractions with the same denominator. By the end of Year 7 students are expected to understand fractions and all fraction operations. Approximately only ten percent (Waters, Lester & Cross, 2014) of Australian students remain at the same school as they transition from primary through to secondary school, usually transitioning between Years 6 and 7, making it difficult to know exactly what experiences secondary students have had with fractions in the past. High school mathematics teachers have limited time in the curriculum dedicated to fraction operations and must assume each student has a good base of conceptual knowledge upon which to build their procedural understanding. Although there is always variation in the teaching methods of the curriculum, this research is based on the assumption that the level of instruction for both fraction concepts and procedures is comparable for each student. This research does not examine initial teaching methods or teacher knowledge. With an assumption that students are, in theory at least, taught the same concepts of fractions and fraction operations, this research seeks to determine what happens when students reach the end of the explicit fraction teaching in the curriculum and still do not understand or remember the procedures?

This thesis is thus concerned with intervention after formal instruction, not a teaching method for delivering new content. After instruction and summative assessments, how do we address the fact that students still do not know how to apply fraction knowledge in the future since they may not have understood the concepts in the first place? Misconceptions arise for a number of reasons and this thesis is concerned with how to “fix” misconceptions rather than on how to teach fractions in their inception.

The research in this thesis pertains to the interference effect of prior learning and the effect "proactive inhibition" has on intervention strategies. Proactive Inhibition is a mechanism that allows us to retain memory but makes it difficult for a person to distinguish between right and wrong information. The higher a person's level of proactive inhibition, the higher the interference effect of prior learning. In a number of domains, a particular approach known as the Old Way/New Way strategy (O/N) has proven to be an effective remediation method to overcome the interference effect. O/N has also been shown to increase confidence in performance, which leads us to ask: Does this method help the learning of fractions? Does this method of remediation increase mathematical self-efficacy?

1.3 Statement of the Problem

Benton J. Underwood first proposed the interference effect of prior learning in the 1960s with the well-known Stroop effect. The Stroop effect is one of the best known phenomena in cognitive science and psychology. In its basic form, the task is to name the colour in which a word is printed, ignoring the word itself. When the word is a colour word printed in a mismatched ink colour, this is very difficult to do and results in slow, error-prone responding (MacLeod, 2015). Harry Lyndon proposed a systematic *unlearning* strategy for overcoming proactive inhibition in the 1980s. This strategy, the Old Way / New Way Technique, has been successful in many fields, such as sport (Hanin, Korjus, Jouste, & Baxter, 2002), workplace training (Weaver, Baxter, & Lyndon, 2000) and in mathematics and science education (Lyndon, 1989; Baxter & Dole, 1990; Henderson, Higgs, Lyndon, Wilkinson, & Yates, 1999).

Fractions, and the problems that arise for school students learning about them, have been discussed for many years. Diagnostic techniques have been identified and utilised by educators worldwide but intervention strategies remain insufficient. Students still 'forget' correct fraction algorithms. Students with low achievement scores also report lower engagement and motivation for mathematics. If we could add all the pieces of information derived from research together, would it be possible to design an effective intervention program for fraction misconceptions that also increased self-efficacy, and therefore motivation, for mathematics?

The aim of the research, therefore, is to test a particular intervention strategy for its effectiveness on fraction misconceptions.

Limits to the research are noted from the start. I do not, for example, examine the students' individual level of proactive inhibition prior to conducting the research. Misconceptions of fraction computation were determined by repeat errors on procedural computations and not conceptual understanding, although conceptual understanding was examined. The O/N technique has successfully been demonstrated in a wide variety of applications where changes in habit, skills, and concepts are required. This implies that quite specific misconceptions of *procedure* in fraction computation were utilised for this research. There was a need for these limitations because the investigation of changes in fraction understanding and remediation and the effect these changes have on self-efficacy is such a broad and complex area and it could not all the examined in one study.

The study is limited to students in one coeducational school due to the difficulty in accessing students for research purposes and for ease of implementing intervention programs twice-weekly over a five-week period.

1.3.1 Aims

This research aims to:

- Examine the difficulties encountered by high school students when operating with fractions;
- Evaluate the effectiveness of the Old Way/New Way (O/N) Technique for addressing systematic error computations in fractions compared with a traditional remediation program;
- Determine if the O/N technique leads to improved fraction computation and understanding; and
- Examine the effects of the two remediation programs on self-efficacy.

1.3.2 Research Questions and Secondary Research Questions

Specifically, the aims allow the investigation of the following Research Questions:

1. What are the difficulties encountered by high-school students when working with fractions?
2. How does the O/N Technique compare to a traditional remediation program when addressing error computations in fractions?
3. What are the effects of the O/N technique and traditional remediation on self-efficacy?

Secondary Research Questions

- What is the relationship between achievement and self-efficacy beliefs?

- In what way do the changes in self-efficacy beliefs differ depending on which intervention program the students participate in?
- What is the relationship between gender and mathematics self-efficacy beliefs?

1.4 Significance

One intended outcome of the study, on a practical level, is to identify students' understanding of fractions and fractions operations, including what procedures they use, and, therefore, what misconceptions are common. Once the level of understanding of fractions is established, a second intended outcome of the study is to identify effective intervention techniques for the remediation of errors. This intended outcome contributes to the design of remedial techniques in the classroom by comparing traditional intervention with the O/N technique. This technique has been used to effectively change habits, skills and concepts in a short period of time in other discipline areas, and has also been used in another sub-domain of mathematics. This leads on to a third intended outcome, namely, expanding the research area of fraction understanding to support the knowledge of what students do, and how to remedy habitual errors in a practical way for increased understanding and increased self-efficacy.

This study will focus on student learning of fractions in Years 7, 8, and 9 of secondary school to determine the effects of two intervention programs. It involves gathering information about achievement on a fraction diagnostic test, the effectiveness of an intervention program on test results, and any resultant changes in mathematical self-efficacy. Any measurable relationships between intervention and achievement, and intervention and self-efficacy will also be examined.

1.5 Overview

The thesis is structured into seven further chapters. Chapters 2 and 3 review the literature relevant to this research. This is separated into two fields: (i) the mathematics-specific areas of fraction understanding and self-efficacy, and (ii) the more general issues of errors, proactive inhibition, and the Old Way / New Way technique. I position the current study in related literature before establishing the research methodology.

Chapter 2 deals with fraction understanding and mathematics self-efficacy and links together the difficulties students face when working with fractions and the effect it has on their beliefs or perceptions with respect to their abilities in mathematics. The concept of a fraction is presented and the issues related to the learning of fractions are discussed in relation to the multifaceted construct of fractions. Chapter 2 concludes with an overview of the literature discussing self-efficacy, and, more specifically, mathematics self-efficacy. The chapter explains, in terms of self-efficacy, the reasons why people engage in tasks in which they feel competent and confident and avoid those in which they do not. I discuss the relationship between cognitive engagement and academic performance, and the influence self-efficacy has on motivation.

In Chapter 3, I discuss the suggestion by Radatz (1980) that errors are the product of previous experiences and are persistent until intervention. Errors are not considered a learning deficit, but a poor transfer, or incorrect construction of learning. A re-teaching strategy does little to fix the problem as old learning disables new learning by a process of psychological interference. In order to explain this phenomenon, I discuss the concept of proactive inhibition, which is an information protection mechanism. Proactive inhibition protects all learned knowledge and skills, right and wrong, and strongly resists and slows down any attempts to change or improve prior knowledge (Underwood 1966). Chapter 3 argues for the need to consider the effect of proactive inhibition on the learning of fractions and the need to investigate ways in which to help students remember correct techniques when in conflict with incorrect information. It looks specifically at the Old Way / New Way technique as a mechanism for doing this. The O/N technique is a strategy that focusses on the erroneous procedure and actively and specifically substitutes it with the correct procedure.

In Chapter 4, the research design is described, providing the philosophical foundation for intervention research using an experimental group design. The chapter describes the theoretical and procedural description of the instruments used in the study to collect, present, and analyse data. This chapter also includes details of the pilot study to test and validate the fractions diagnostic test prior to the research being undertaken.

Chapters 5, 6, and 7 are results chapters. In Chapter 5, the results about students' fraction understanding are presented. In Chapter 6, results of the intervention programs are presented. Finally, in Chapter 7, results and analysis of the self-efficacy questionnaires are presented. Each individual results chapter provides a discussion of the results in relation to the aim of the chapter and the relevance to theories and literature for each topic.

In Chapter 8, conclusions from the research are drawn. The results of this study are discussed in relation to the research aims listed in Chapter 1. Limitations are outlined, and implications for further research are proposed.

Chapter 2

Literature Review: Fractions and Self-efficacy

2.1 Introduction

The complexities of fraction understanding have been researched and discussed for many years. Teaching fractions for understanding was recommended by Kieren (1976) when he was the first to suggest that fractions be conceptualised as a set of interrelated constructs. Research (e.g. Byrnes & Wasik, 1991) then suggested the need to teach conceptual understanding before procedural. Forty years later and the research still highlights the difficulties fractions present to both teachers and students. Why are we still having the same problems? Despite many studies highlighting the common misconceptions and outlining developmental suggestions, students are still encountering difficulties in grasping the concept of fractions. The necessity to master fractions has been firmly established and to support this view there has been a lot of work done to design constructs, recommend developmental stages, and diagnose misconceptions. Despite this, however, students are still forming misconceptions. If they carry these through to the high school years, it is important to ask how to approach intervention so that students are not missing the knowledge required to progress their mathematical understanding.

The first purpose of this chapter is to examine the literature pertaining to the difficulties students have in learning fractions and the misconceptions that result from not understanding fractions. In order to discuss errors in fraction understanding, I will distinguish between careless mathematical errors and those that are repeated erroneous misconceptions. I will examine more closely the concept of a fraction, how fractions are taught, the development of fraction knowledge, and where fractions are positioned within the Australian Curriculum. I will discuss the methods used for diagnosing fraction misconceptions and examine the intervention methods currently used in educational settings to enable students to better understand both conceptual and procedural aspects of fraction learning.

The second purpose of this chapter is to examine the role self-efficacy plays in the learning process. I will examine the links between self-efficacy, motivation, and engagement, and will review the literature on mathematics self-efficacy.

2.2 Errors in Mathematical Algorithms

Research into students' mathematical errors has significantly influenced the areas of mathematics assessment and intervention, providing insight into common *error techniques* (Ashlock, 1990, 1994; Borasi, 1987, 1994 & Radatz, 1980). To separate errors and misconceptions, it is useful to consider the definitions used by researchers to differentiate the two. Hawker and Cowley (1998) defined errors as "mistakes or a condition of being wrong" (p. 163) and associated errors with performance that is evaluated after instruction. Misconceptions differ in that students conceptualise a belief within themselves before instruction begins. According to Bell (1984) a misconception is "… the implicit belief held by a pupil, which governs the errors that pupil makes" (p. 58). Rowell, Dawson, and Lyndon (1990) described misconceptions as reliable aspects of an individual's theoretical knowledge, that are made evident by the consistent use of a false explanation. Misconceptions are constructions that start with the recognition of a knowledge "gap", which continues depending on what the individual already knows and on the range and sequence of experiences faced. According to Rowell, Dawson, and Lyndon the resulting knowledge, the misconception, is the "best" the individual can manage to produce at that point in time, "and is the starting point for any further progress, irrespective of whether it's limited or wrong" (p. 168).

Perso (1992) described the relationship between misconceptions and errors as errors resulting from misconceptions: "… errors are not simply failures by students but rather symptoms of the nature of the conceptions which underlie their mathematical actions" (p. 349). In this context, Perso is suggesting that errors are the signifier that there is some underlying conceptualisation in the students' schema that is incorrectly or incompletely constructed, which other researchers would describe as misconceptions. Perso suggested that incorrect answers could be a result of guessing or low mathematical aptitude (errors) but more often they result from systematic strategies or rules that have sensible origins and are based on beliefs (misconceptions).

Olivier (1989) described misconceptions as being erroneous thinking that students consistently apply. This was supported by Baroody and Hume (1991) when they demonstrated that students' errors in using algorithms were often not caused from failing to learn a particular concept, but rather from learning or constructing incorrect mathematical ideas. In a chapter relating to diagnostic teaching and professional judgement, Tripp (1993) stated that "Students do misunderstand, but it is seldom because they cannot understand, most often it is because they understand something else" (p. 88). As Rowell, Dawson, and Lyndon (1990) suggested, the important feature of misconceptions is that misconceptions are knowledge, and for the student who holds the misconception, this knowledge is no different to any other knowledge they have constructed.

Piaget's (1971) concept of *assimilation* helps us understand why misconceptions persist. As Longfield (2009) suggested, new information is more easily learned when it can be related to something that is already known (existing schema). Assimilation describes the type of learning that occurs when information can be taken on board without revisiting cognitive frameworks. This is in contrast to *accommodation*, which describes how we must revise what we already know (or thought we knew) to accommodate a new idea. Longfield described how we have a natural tendency to overemphasise information that supports our current theories and discount information that would throw us into disequilibrium. Once a schema or concept is formed, it is stable and resistant to change. A student's existing schema will therefore determine what that student will then learn from experience or instruction.

Ashlock (2010) suggested that algorithms incorporating error patterns are often referred to as "buggy algorithms" (p. 9). A buggy algorithm includes at least one erroneous step, and the procedure does not consistently accomplish the intended purpose. Steinle (2004) commented that:

> It is helpful for teachers to know that misconceptions and buggy errors do exist, that errors resulting from misconceptions or systematic errors do not signal recalcitrance, ignorance, or the inability to learn; how such errors and misconceptions and the faulty reasoning they frequently signal can be exposed; that simple telling does not eradicate students' misconceptions or "bugs" and that there are instructional techniques that seem promising in helping students overcome or control the influence of misconceptions and systematic errors. (pp. 1-2)

Mestre (1989) described two reasons why misconceptions are a problem; firstly, they interfere with subsequent understandings if the student attempts to use them as a basis for further learning. Secondly, they have been actively constructed by the student and therefore have emotional and intellectual attachment for that student, and consequently are only relinquished by the student with reluctance. If misconceptions are as pervasive and resilient as they are reported, the question arises as to whether they can be addressed and alleviated through intervention.

As prior knowledge has been established as an important aspect of teaching strategies, Rowell and Dawson (1988) suggested that when students find new information to be in conflict with prior spontaneous beliefs or knowledge they will not readily accept it. They proposed that the construction of new knowledge, and belief, should be on the basis of the old. They proposed in that way "… the previously unbelievable becomes believable, and only when that has occurred does the teacher bring the new beliefs into conflict with the old …" (p. 150). In collaboration with Lyndon (Rowell, Dawson, & Lyndon, 1990) they further proposed that to optimise the rate of progress in overcoming a misconception, students should first construct another relevant, potentially contradictory, better explanation. "This explanation should be firmly based on aspects of their knowledge structure which, although not spontaneously used, are familiar to them" (p. 168). Their rationale for this approach was that an individual would be in a better position to rationally argue which theory to retain if a new and intrinsically better theory had been constructed.

Based on the fact that learning is not linear, rather a progression through a series of understandings and misunderstandings, Bell (1984) suggested that, if they arise, teachers should embrace misconceptions as an important stage of learning. This study will therefore concentrate on the misconception's students have when working with fractions and aims to find an effective intervention method to alleviate these misconceptions based on the idea of bringing new beliefs in conflict with the old.

2.3 The Difficulties and Errors in Learning Fractions

2.3.1 The Concept of Fractions and the Development of Students' Knowledge

Fractions are the first place in which children encounter expressions, like $\frac{3}{4}$, that represent relationships between two discrete or continuous quantities, proportions, or parts of a whole. Experience with these concepts begins early, before formal schooling and extends into the high school years. Before they learn anything formal in classrooms, students engage in activities in their everyday life where they generate ideas about fractions. In the 1990s there was renewed interest in the role that cultural processes play in the understanding of mathematics and in the performance of mathematics tasks. Rogoff (1990) and Steffe, Cobb, and von Glaserfeld (1988) believed language to be one of the most important factors in these processes. The language of mathematics provides a cultural context for mathematical activities and may make some tasks easier to learn.

In a study that examined children's knowledge of numerical fractions prior to school instruction, Miura, Okamoto, Vlahovic-Stetic, Kim, and Han (1999) demonstrated that children's cognitive representation of number was influenced by the structure of their number-naming systems. They suggested that certain characteristics of Asian number language promoted a developmental head start and affected the later performance of mathematical tasks. In Korean, Chinese, and Japanese, the concept of fractional parts is embedded in the mathematics terms used for fractions. For example, one fourth is spoken as "of four parts, one". The oral term expresses the part-whole relation and may influence early conceptualisation of fractions. Nunes (1992) suggested that with experience comes understanding and when the whole-part relation is an integral part of the linguistic terminology, children are able to make sense of the relation between fraction terms and their visual representations. Miura, Okamoto, Vlahovic-Stetic, Kim, and Han (1999) suggested that once the unit fraction is understood, knowledge can easily be extended to more complex fractions. This language difference is explicitly referred to in the Australian Curriculum in the Year 3 elaboration of modelling and representing unit fractions and their multiples to complete a whole, "recognising that in English the term 'one third' is used (order: numerator, denominator) but that in other languages this concept may be expressed as 'three parts, one of them' (order: denominator, numerator), for example Japanese" (ACARA, 2017).

Smith (2002) suggested that another way young children develop ideas about rational numbers is in a variety of real-life situations, such as measuring and dividing continuous quantities and quantitative comparison of two quantities. They construct knowledge about relational numbers and bring this *constructed* knowledge into the classroom where it interacts with the curriculum (*instructed knowledge*). Smith believed mathematically successful students manage to connect these two bodies of knowledge and students who never really understand do not.

Pothier and Swada (1983) documented that students come to instruction with informal knowledge about partitioning and equivalence. Behr, Wachsmuth, Post, and Lesh (1984) found that children come to instruction with informal knowledge about joining and separating sets and estimating quantities involving fractions. Ball (1993) claimed that learning mathematics with understanding entails making connections between informal understandings and more formal mathematical ideas. Mamede, Nunes, and Bryant (2005) suggested that more research was needed to explore how students build upon their informal knowledge to improve their understanding of fractions.

Smith (2002) generalised that children's knowledge of fractions moves through two broad phases of development: making meaning for fractions by linking quotients to divided quantities, and exploring the mathematical properties of fractions as numbers. Students must understand the key idea that fractions name the relationship between the collection of parts and the whole, not the size of the whole nor its parts. For students to succeed, they must understand they are dealing with relative amounts, not absolute size or amount. Once students can generate different continuous or discrete quantities for fractions, they are ready to explore fractions as a system of numbers. Some of the most challenging of the necessary concepts include equivalence and order properties.

Between 2005 and 2014 over one hundred research reports on the difficulties of fractions were presented to the International Group for the Psychology of Mathematics Education (PME), highlighting different aspects of understanding and operating with fractions. Zazkis and Mamolo (2016) found that one of the main difficulties highlighted in the research related to the fact that students encounter fractions after they have established ideas and procedures for natural numbers. These earlier experiences may influence the student's expectations for working with fractions, a phenomenon referred to as “natural number bias”. Obersteiner, Van Hoof, Verschaffel, and Van Dooren (2016) stated that while students’ prior knowledge of natural numbers is essential for learning rational numbers, it can cause systematic mistakes on tasks in which this knowledge is not applicable. For example, when students compare the numerical value of fractions, they often consider a fraction as two natural numbers rather than one holistic number and may suggest that $\frac{1}{4} > \frac{1}{3}$ because $4 > 3$. Students also have a prior knowledge about the effects arithmetic operations have on numbers. For natural numbers, addition and multiplication (by a number other than 1) always makes a number bigger, whereas subtraction and division always make a number smaller. This is not always true, however, for the effect of arithmetic operations of rational numbers smaller than 1 or less than zero.

Van Hoof, Vendewalle, and Van Dooren (2013) found that students were more often correct when natural number knowledge led to correct answers, that is, the use of natural number knowledge led to a correct conclusion. This was in comparison to when the bias led to a wrong conclusion. Examples of their experimental items where intuitive reasoning led to the correct answer consisted of fractions of the same denominator ($\frac{a}{x}$ and $\frac{b}{x}$): in this case, if a >b, then $\frac{a}{x} > \frac{b}{x}$. Instances where the natural number bias led to incorrect answers were in questions where the fraction had equal value numerators but differing denominators ($\frac{x}{a}$ and $\frac{x}{b}$) and the intuitively generated response was that if a > b, then $\frac{x}{a} > \frac{x}{b}$. This finding was supported by Gomez, Jimenez, Bobadilla, Reyes, and Dartnell (2014) who explored the extent to which natural number bias accounted for the errors made by students in a fraction comparison questionnaire. One quarter of the participants responded with “extreme bias”, in that they were 100% accurate on items where natural number knowledge led to correct answers and made errors on all items in which the natural number bias would give incorrect answers. Obersteiner, Van Hoof, Verschaffel, and Van Dooren (2016) concluded that teachers should make all students aware that they have intuitions about natural numbers that may be misleading when working with rational numbers.

Prediger (2006) described the natural number bias in fraction learning as a “discontinuity”. She described early understandings of natural numbers as barriers to the construction of new understanding and that “… students see continuities where discontinuities in dealing with numbers should appear” (p. 377). She highlighted the need for focussing on the precise location of students’ difficulties with discontinuities in the learning process. Prediger provided a conceptual tool, differentiating between intuitive, algorithmic, and formal understanding between natural and fractional numbers (see Figure 2.1).

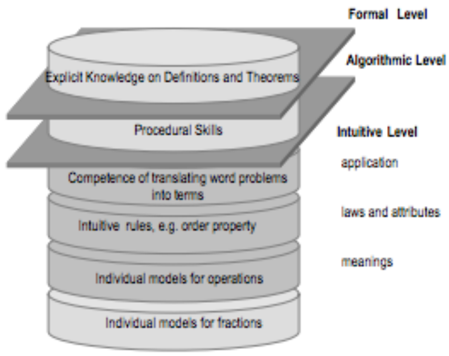


Figure 2.1. Conceptual tool for locating difficulties with discontinuities (Prediger, 2006, p. 378).

In Figure 2.1, the intuitive level is characterised as the type of implicit knowledge that is confidently accepted as being obvious. The algorithmic level of knowledge is procedural in nature, and involves students' capability to explain the successive steps in procedural operations. The formal level includes the definitions of concepts and of operations, structures, and theorems relevant to a specific content domain. Prediger believed nearly all studies focusing on conceptual change in the field of fractions have treated discontinuities at the intuitive knowledge level, but have focused on *rules*, neglecting the sub-level of *meanings*. Her findings demonstrated that all levels were highly connected, with each level giving reasons for obstacles in the upper level. She concluded that the transfer of rules from natural numbers to fractions appeared to be a problem of generalisation.

There is consensus among researchers that another predominant factor contributing to the complexities of teaching and learning fractions lies in the fact that fractions are a multifaceted construct (Brousseau, Brousseau, & Warfield, 2004; Kieren, 1995; Lamon, 2001). Kieren (1976) was the first to propose a multifaceted construct of fractions. Kieren initially identified four sub-constructs, with the notion of the part-whole relationship considered the basis for development of the other sub-constructs. Kieren claimed that the part-whole notion was embedded within all the other subsconstructs and therefore did not identify this concept as separate, fifth, sub-construct. The *part-whole* interpretation of rational number was considered by Kieren (1981) to be an important language-generating construct, which depends directly on the ability to partition either a set of discrete objects or a continuous quantity into equal-sized subparts. The *ratio* sub-construct of rational number expresses a relationship between two quantities. The fraction as *operator* sub-construct suggests a rational number is a transformation. The *quotient* sub-construct interprets a rational number as an indicated quotient, i.e., $\frac{a}{b}$ is interpreted as a divided by b. The fractional *measure* sub-construct represents a reconceptualisation of the part-whole notion of fraction. It highlights how much there is of a quantity relative to a specified unit of that quantity.

According to Clarke, Roche, and Mitchell (2008) the important *part-whole* interpretation of rational number depends on the ability to partition either a continuous quantity or a set of discrete objects into equal sized subparts or sets. Mamede, Nunes, and Bryant (2005) described part-whole situations as the denominator designating the number of parts into which a whole has been cut and the numerator designating the number of parts taken. Lamon (1999) explained that the *measure* interpretation is different from the other constructs in that the number of equal parts in a unit can vary depending on how many times you partition. Clarke, Roche, and Mitchell state that the successive partitioning allows you to measure with precision. These measurements can be represented by 'points' on a number line. A fraction may also represent the operation of division or the result of a division such as $2 \div 3 = \frac{2}{3}$. The division or *quotient* may be understood through portioning and equal sharing. The denominator designates the number of recipients and the numerator designates the number of items being shared. Mamede, Nunes, and Bryant describe the difference by using $\frac{2}{4}$ as an example. The part-whole situation means that a whole was divided into four equal parts, and two were taken. In a quotient situation, $\frac{2}{4}$ means that 2 items were shared among four people. In a measure situation $\frac{2}{4}$ means one whole (of a line segment, for example) has been divided into 4 equal parts and two units of measure are expressed.

A fraction can be used as an *operator* to operate on a unit, such as $\frac{3}{4}$ of 8 = 6. The misconception that multiplication "always makes bigger" and division "always makes smaller" is common (Clarke, Roche, & Mitchell, 2008). Fractions can also be written as a *ratio*, which expresses the relationship between numbers of the same kind.

Behr, Lesh, Post, and Silver (1983) further developed Kieren's ideas, recommending that the part-whole relationship comprise a distinct sub-construct of fractions, also connecting this with the process of portioning. They went on to propose a theoretical model linking the different interpretations of fractions to the basic operations of fractions and to problem solving (Figure 2.2).

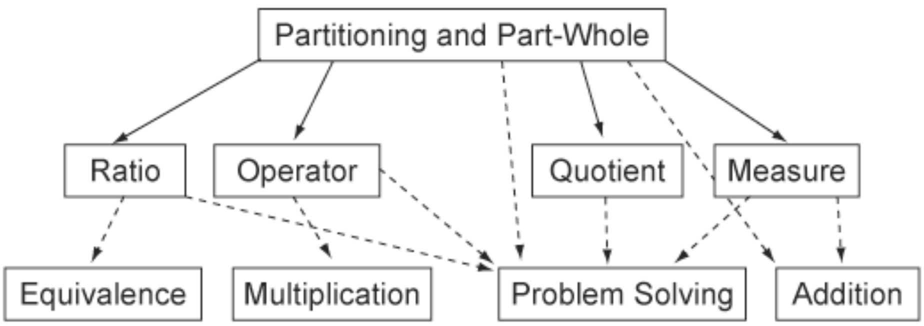


Figure 2.2 Behr, Lesh, Post and Silver's (1983) adapted model of fraction sub-constructs.

Behr, Lesh, Post, and Silver's model provides support to the assumption that mastering the interpretations of fractions contributes towards acquiring proficiency in the operations of fractions. This finding was attributed to the fact that students' performance on the operations of fractions required both procedural fluency and conceptual understanding of the operations (Charalambous & Pitta-Pantazi, 2005). The Behr, Lesh, Post, and Silver model (1983) illustrates the importance of the part-whole/partitioning sub-construct, which they consider to be fundamental for developing understanding of the other four sub-constructs of *ratio*, *operator*, *quotient*, and *measure*. Their model suggests the concept of ratio should be used to promote equivalence and the process of finding equivalent fractions. The operator sub-construct, once understood, can promote multiplication, and measure is the most natural pathway to develop understanding in the additive operations of fractions. Understanding of all five of these sub-constructs is considered a prerequisite for fraction problem solving.

Mack (2001) proposed a different classification using partitioning to cover both part-whole and quotient situations. Despite slight variations, part-whole, quotient, measures, and operator situations are common to all of them. In reviewing the theoretical models of fractions, Charalambous and Pitta-Pantazi (2005) provided support to the assumption that mastering the five interpretations of fractions: part-whole, ratio, operator, quotient, and measure, contributes towards acquiring proficiency in the operations of fractions. They suggested that the teaching of fractions needed to be scaffolded to develop a profound understanding of the different interpretations of fractions. Specifically, the findings of their study provided empirical support to the fundamental role of the part-whole sub-construct in building understanding of the remaining constructs of fractions. This supported previous studies, (e.g. Lamon, 1999; Brousseau et al., 2004) that suggested emphasis should be placed on the conceptual understanding of fractions and the teaching of the different operations should be directly linked to specific interpretations of fractions. Reys et al. (2012) suggested that students acquire and develop misconceptions when they have difficulty incorporating procedural instruction into their conceptual framework, concluding that students should not learn procedures too early; rather, students should start by using their understanding of fractions to develop procedures that make sense to them.

2.3.2 Conceptual and Procedural Understanding

Many different terminological frameworks have been used in mathematics teaching and learning literature over the past century to describe knowledge, including meaning theory (Brownell, 1945), relational understanding (Skemp, 1976), and routine and adaptive expertise (Hatano & Inagaki, 1986). Since the 1980s, however, the most prevalent of these frameworks is one identifying two major kinds of knowledge, *conceptual knowledge* and *procedural knowledge* (Hiebert & Lefevre, 1986). Hiebert and Lefevre's work provided definitions of conceptual and procedural knowledge. *Conceptual knowledge* is typically defined as

> knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which linking relationships are as prominent as the discrete pieces of information. Relationships

> pervade the individual facts and propositions so that all pieces of information are linked to some network. (pp. 3-4)

Procedural knowledge is defined in terms of two kinds of knowledge:

> One kind of procedural knowledge is a familiarity with the individual symbols of the system and with the syntactic conventions for acceptable configurations of symbols. The second kind of procedural knowledge consists of rules or procedures for solving mathematical problems. Many of the procedures that students possess probably are chains of prescriptions for manipulating symbols. (Hiebert & Lefevre, 1986, pp. 7-8)

In a 2009 report, the National Mathematics Advisory Panel (in the USA), using Hiebert and Lefevre's definitions of conceptual and procedural understanding, suggested that learning mathematics requires three types of knowledge: factual, procedural, and conceptual. Factual knowledge refers to a situation where information or answers do not need to be calculated but simply retrieved from memory. Procedural knowledge is knowing the sequence of steps to solve a frequently encountered problem and assumes a foundation of factual knowledge. Conceptual knowledge refers to an understanding of meaning, of knowing *why* a calculation works, and the facts are no longer isolated but become organised in coherent structures.

Wong and Evans (2007) suggested that a mathematical concept is not a single isolated idea but one idea in a structured system of knowledge and that conceptual understanding is intertwined with procedural knowledge. They believed that investigation into a student's knowledge of a mathematical concept required more than a determination of correctness/incorrectness. It required further investigation into the response, which can provide valuable insight into the thinking (Gould, 2005).

One explanation for student's difficulties when learning fractions lies in the articulation between conceptual and procedural knowledge. Huinker (2002) surmised that the traditional teaching approach for fractions has been heavily symbolic and procedural. The rush to show students how to perform procedures prevents them from establishing a proper understanding of a concept. She suggested a shift from learning computational rules to developing fraction operation sense. Fundamental to operation sense is an understanding of the meanings and models of operations. Huinker presented seven dimensions of operation sense: understanding the meanings and models of operations, understanding the effects of an operation on a pair of numbers, real-world problems, understanding the meaning and mathematical language associated with operations, ability to translate across various models of interpretation, understanding relationships among operations, and the ability to compose and decompose numbers and use properties of operations to solve mathematical problems.

Huinker (2002) conducted research with Year Five students to see how they defined and characterised number and operation sense of fractions. Huinker revealed initially the students overwhelmingly manipulated symbolic representations with little understanding. Over a four-week instructional period, students made significant progress in their fraction operation sense even though they were not taught procedures for solving fraction computation problems.

Byrnes and Wasik (1991) conducted two studies with seventy students from Years 4, 5 and 6 that suggested that students gain conceptual understanding before they gain procedural competence with fractions, and, further, that many of the conceptual aspects of fractions are prerequisites for the procedural ability to perform computations. In a test of fraction equivalence and fraction addition, where the addition of fractions with different denominators required the same type of knowledge the students used to determine equivalence, the results reported students used conceptual knowledge before procedural. For example, in fraction comparison questions the students in the Byrnes and Wasik study were able to demonstrate other ways in which they could compare fractions without using the common denominator procedure. From these results, Byrnes and Wasik concluded that conceptual understanding precedes procedural knowledge.

The conclusion from the Byrnes and Wasik (1991) studies supported the view that Rittle-Johnson, Siegler, and Alibali (2001) calls the "concepts-first" approach. The theoretical assumption on which this view is based is that conceptual understanding is the driving force of cognitive development. We learn only by understanding, and procedural knowledge is only the set of helpful tools we employ once we have conceptual knowledge (Hallett, Nunes, & Bryant, 2010). If the procedures are not understood conceptually, they will be prone to the development of "bugs" (Brown & Van Lehn, 1982; Resnick, 1983; Van Lehn, 1982). Examination of the errors that students make suggests that they make many errors when they use a procedure without conceptual understanding (e.g., Bezuk & Cramer, 1989; Kerslake, 1986; Skemp, 1976).

In their study of individual differences in conceptual and procedural knowledge when learning fractions, Hallett, Nunes, and Bryant (2010) hypothesised that there would be individual differences in the way students learnt. In a study involving Year 6 (n=119) and Year 8 (n=114) students they provided evidence that some students relied more on concepts, some relied on procedural knowledge, and some relied on both. For example, the students in their study were asked to answer the the question $\frac{1}{2}+\frac{1}{4}$, some could do so conceptually by understanding that there are 2 quarters in a half and then adding all the quarters. These students demonstrated their conceptual knowledge of portioning. When given the same question, others approached it procedurally by applying the lowest common denominator algorithm. Some students also tried a combination of the approaches, demonstrating a reliance on both. Hallet, Nunes, and Bryant also found that there were two types of students who struggle with fractions: one group that had problems with conceptual knowledge and one group that had problems with procedural knowledge. These findings supported the work of Kerslake (1986) and Peck and Jencks (1981) where students were able to correctly apply procedure without understanding why the procedures worked. Although the work of Rittle-Johnson et al. (2001) suggested that learning conceptual knowledge helps a student learn procedural knowledge, it does not consider individual difference and does not take into account that some students might rely more on one type of knowledge than the other (Hallett, Nunes, & Bryant, 2010).

2.3.3 The Complexities of Fraction Understanding

Various studies have considered the existence of interrelated fraction concepts as a major factor contributing to the difficulty of developing fraction understanding (Behr, Khoury, Harel, Post, & Lesh, 1997; Behr et al., 1983; Charalambous & Pitta-Pantazi, 2006; Siemon et al., 2015). Getenet and Callingham (2017) used Behr et al.'s (1983) model to investigate practices in a New Zealand high school class. Their study showed that the most frequently observed fraction concept reflected in the teacher's and students' discussions, and their use of language, was the part-whole concept, whereas measure arose the least. Siemon et al. (2015) suggested that fraction concepts are often taught using procedures and memorisation, rather than having students develop their own understanding.

Bezuk and Cramer (1989) suggested the way in which fractions are taught must be changed and proposed a shift from the development of algorithms for performing operations on fractions to the development of quantitative understanding. In Australia, fraction concepts are introduced as early as Year 1, though the main work on them begins in Year 4. Bezuk and Cramer's recommendations included the use of manipulatives as crucial in developing fraction ideas, allowing students to construct mental referents that enable them to perform fraction tasks meaningfully. They suggested that the majority of instructional time before Year 6 should be devoted to the development of concepts and relationships. Operations on fractions, in contrast, should be delayed until concepts and the ideas of the order and equivalence of fractions are firmly established, and the size of the denominators in computational exercises should be limited to numbers 12 and below.

The Queensland Curriculum and Assessment Authority (QCAA, 2013) examined students' National Assessment Program – Literacy and Numeracy (NAPLAN) results for fraction-related concepts within the *Number* sub-strand: including part-whole relations, equivalence, and percentages. Data from the NAPLAN testing found that the facility rates for problems involving fractional concepts were low, indicating that some students struggle with the basic concepts of part-whole relationship and equivalence. The QCAA reported that the lack of conceptual understanding would compound students' difficulties in developing higher-order thinking, such as proportional reasoning. Findings for Years 7 and 9 showed that students struggled with the concept of parts of the whole, with fundamental systematic errors across both year levels. Students who could not identify the whole had incorrect responses on successive calculations. The gaps in students' conceptual understanding were reported as underdeveloped procedural knowledge, which hindered their application of mathematical content knowledge to check the reasonableness of their answers. The report provided examples of how students' underdeveloped understanding of the part-whole concept affected their performance.

Queensland Year 7 and Year 9 students performed below the national average in a question where they were required to express the part of green apples as a fraction of the whole collection (24 red + 12 green), with only 47.5% of Year 7 students and 57.3% of Year 9 students answering the item correctly. The most common error was the answer of $\frac{1}{2}$, illustrating that some students did not identify the appropriate whole ($24 + 12 = 36$), instead choosing the denominator to be larger of the two numbers, 24. These students, whose understanding of the whole was incomplete, were more likely to find the concept of equivalence challenging. This was supported in the findings that 44% of Year 7 students had difficulty with the concept of equivalence.

A question on proportional reasoning (see Figure 2.3) demonstrated that nearly 60% of students in Year 7 and 50% of Year 9 students could not demonstrate their knowledge of the concept of percentages in a multistep problem with whole numbers and a common fraction. Students without part-whole understanding and understanding of equivalence will struggle with problems involving multiplicative reasoning and to interpret equivalent representations of fractions.

A copier prints 1200 leaflets.
One-third of the leaflets are on yellow paper and the rest are on blue paper.
There are smudges on 5% of the blue leaflets.
How many blue leaflets have smudges?

40	60	400	800
○	○	○	○

Figure 2.3 Example NAPLAN question on proportional reasoning

The majority of incorrect responses were the second option, "60" likely arising from having simply computed 5% of the given 1200 and not recognising they had to calculate the number of blue leaflets and then find $\frac{5}{100}$ of the blue leaflets. Incorrect responses suggest an inability to identify the parts of the whole and/or an inability to understand the relationships among the components in the problem. A similar question in the following year, in which an item involved a simple estimate of a common fraction from a given proportion involving whole numbers, reported facility rates of 32% in Year 7 and 37% in Year 9, suggesting that understanding of this concept is not prevalent.

The QCAA report concluded that the learning of fractions is difficult. Gaps in learning appear to be overlooked, leading to students being introduced to the more difficult concepts before they understand earlier ones. This hinders students' understanding and limits their ability to solve problems relating to these concepts. The QCAA suggests that revisiting fundamental concepts to develop understanding have proved to be successful. This supports Lesh, Post, and Behr's (1988) conclusion that understanding equivalence is a central component of proportional reasoning and Behr, Waschmuth, Post, and Lesh's (1984) finding that even after learning fractions for many years, many students still struggle with the concept of equivalence and development can only occur by revisiting concepts and gaining competence.

2.3.4 Fractions in the Australian Curriculum

The Australian Curriculum, Assessment and Reporting Authority (ACARA) places emphasis on fractions across all year groups, starting with recognition of "one-half" as early as Year 1. Operations with fractions begin in Year 5 and by the beginning of Year 8 students should be competent in all fraction operations and should be able to connect and convert among fractions, decimals, and percentages. Work with fractions after Year 8 is focused on operations with simple algebraic fractions and the ability to solve linear equations involving simple algebraic fractions. Table 2.1 provides specific content descriptors from Australian Curriculum: Mathematics (ACARA, 2017).

Table 2.1

Content descriptors related to fractions from the Australian Curriculum: Mathematics *(ACARA, 2017).*

Year	Fractions in fractions and decimals, and real numbers	Content Descriptor
1	Recognise and describe one-half as one of two equal parts of a whole.	ACMNA016
2	Recognise and interpret common uses of halves, quarters and eighths of shapes and collections	ACMNA033
3	Model and represent unit fractions including 1/2, 1/4, 1/3, 1/5 and their multiples to a complete whole	ACMNA058
4	Investigate equivalent fractions used in contexts	ACMNA077
	Count by quarters, halves and thirds, including with mixed numerals. Locate and represent these fractions on a number line	ACMNA078
	Recognise that the place value system can be extended to tenths and hundredths. Make connections between fractions and decimal notation	ACMNA079

5	Compare and order common unit fractions and locate and represent them on a number line	ACMNA102
	Investigate strategies to solve problems involving addition and subtraction of fractions with the same denominator	ACMNA103
6	Compare fractions with related denominators and locate and represent them on a number line	ACMNA125
	Solve problems involving addition and subtraction of fractions with the same or related denominators	ACMNA126
	Find a simple fraction of a quantity where the result is a whole number, with and without digital technologies	ACMNA127
	Make connections between equivalent fractions, decimals and percentages	ACMNA131
7	Compare fractions using equivalence. Locate and represent positive and negative fractions and mixed numbers on a number line	ACMNA152
	Solve problems involving addition and subtraction of fractions, including those with unrelated denominators	ACMNA153
	Multiply and divide fractions and decimals using efficient written strategies and digital technologies	ACMNA154
	Express one quantity as a fraction of another, with and without the use of digital technologies	ACMNA155

	Connect fractions, decimals and percentages and carry out simple conversions	ACMNA157
	Recognise and solve problems involving simple ratios	ACMNA173
8	Carry out the four operations with rational numbers and integers, using efficient mental and written strategies and appropriate digital technologies	ACMNA183
	Solve a range of problems involving rates and ratios, with and without digital technologies	ACMNA188
10	Apply the four operations to simple algebraic fractions with numerical denominators	ACMNA232
	Solve linear equations involving simple algebraic fractions	ACMNA240

Behr et al. (1983) proposed that when the five sub-constructs for rational number are combined together they create a generalised understanding of fractions. Behr et al., suggested that part-whole comparisons and partitioning are the core concepts that inform and influence understanding of ratio, operator, quotient and measure. The *whole-part / partitioning* sub-construct is introduced into the Australian Curriculum early, with Year 1 students asked to recognise and describe one-half as one of two equal parts. This includes the whole as both continuous (splitting an object into two equal pieces and describing how the pieces are equal) or discrete (sharing a collection of readily available materials into two equal portions). In Year 3 the concept of partitioning is explored with students expected to be able to model and represent unit fractions and their multiples to complete a whole. The elaborations for this descriptor include "partitioning areas, lengths and collections to create halves, thirds, quarters and fifths, such as folding the same sized sheets of paper to illustrate different unit fractions and comparing the number of parts with their sizes" (Australian Curriculum, Year 3 Mathematics Content Descriptions).

The *operator* sub-construct is included in the Year 6 curriculum with students expected to be able to find a simple fraction of a quantity where the result is a whole number. This is followed by multiplication and division of fractions in Year 7, which, according to the model, will be better understood if the concept of operator is learnt. The *quotient* sub-construct is not explicitly referred to until Year 7 in the Australian Curriculum. Students are expected to be able to divide fractions using efficient written strategies. Problem solving requires all five sub-constructs to be understood. If a student does not have practice with partitive division then it would be difficult to solve sharing problems, such as "three children share eight biscuits equally. How much biscuit does each child get?" In this example, the quotient refers to partitive (sharing) division in which the quotient is a fraction of a referent whole or one. Quotitive division, or measurement division, is also important and should distinguished from the sharing interpretation beginning in Year 5 (Wright, Tabor, & Ellemor-Collins, 2011).

All operations of fractions, including ordering, addition, and subtraction, involve the *measure* sub-construct and are usually modelled as a unit of comparison. In the Australian Curriculum, students are introduced to the concept of fractions as measure in Year 5. In the content description

ACMNA102 students are expected to compare and order common unit fractions and locate and represent them on a number line. This is in reference to a continuous context and does not include the measure construct in discrete contexts.

According to the Kieren-Behr model, the *ratio* construct assists with the understanding of *equivalence*, yet ratio is not referred to until Year 8 on the Australian Curriculum whereas *equivalence* is introduced in Year 4.

2.3.5 The Importance of Learning Fractions

Children encounter fractions and fraction-related concepts in real-life and classroom situations, and a firm understanding of fractions undoubtedly helps children make sense of a number of other ideas in their daily life (Yusof & Malone, 2003). The meaning of fractions is part of everyday life and is used in situations such as the estimation of rebates, following a recipe, or reading a map (Gabriel et al., 2013). For example, the numerical scale of a map is often written as a fraction, i.e. 1:50 000 means that one unit of measure on a map is equal to 50 000 units of the same measure on the ground. Fractions also provide students with important prerequisite conceptual foundations for the growth and understanding of other number types and algebraic operations in the later years of mathematics. Specifically, they are involved in probabilistic, proportional, and algebraic reasoning. Bailey et al. (2012) also described the strong predictive relation between earlier knowledge of fractions and later mathematics achievement. Fractions are, however, among the most complex mathematical concepts that children encounter and, traditionally, teaching and learning fractions has been problematic (Newstead & Murray, 1998).

Understanding the difficulties of learning fractions is crucial as these difficulties may affect opportunities for further engagement in mathematics and may lead to mathematics anxiety (Gabriel et al., 2013).

Siegler et al. (2012) examined long-term predictors of high school students' performance with algebra and overall mathematics achievement. Analyses of longitudinal data sets from across the USA and UK revealed that primary school students' knowledge of division and fractions uniquely predict students' performance with algebra and overall mathematics achievement in high school. Siegler, Thompson, and Schneider (2011) proposed the theory of numerical development, a process of progressively broadening the class of numbers that are understood to possess magnitudes. One implication of this theory is that acquisition of fractions knowledge is crucial to numerical development. Fractions provide the first opportunity to learn that several salient and invariant properties of whole numbers are not true of all numbers, for example that multiplication does not necessarily produce answers greater than the multiplicands. According to Siegler et al. (2012) if students do not understand fractions, they cannot estimate answers to simple algebraic equations. Students who do not understand fraction magnitudes also would not be able to reject flawed solutions by reasoning that the answers yielded are impossible. The findings from their investigation demonstrated that primary school students' knowledge of fractions and whole-number division predicted their mathematics achievement in high school, above and beyond the contributions of their knowledge of whole-number addition, subtraction, and multiplication; verbal and nonverbal IQ; working memory; family education; and family income. Their analyses also showed that the predictive strength of early knowledge of fractions and division did not differ between students with greater and lesser mathematics achievement in high school. They concluded that the unique predictive value of early fractions knowledge seemed to be due to many students not mastering fractions and division and to those operations being essential for more advanced mathematics, rather than simply to fractions and division being relatively difficult to master. Siegler et al. highlighted how, over 30 years of nationwide standardised testing in the USA, the mathematics scores of high school students have hardly changed, and suggested that mastery of fractions and division was needed if substantial improvements in mathematics performance are to be achieved.

Pearn and Stephens (2017) reported on the link between fractional knowledge and readiness for algebra. Their research demonstrated that fractional knowledge appeared closely related to establishing algebra knowledge in the domain of solving linear equations. They found that students who relied on visual methods or additive methods experienced difficulty in adopting a multiplicative approach. They concluded that those students were more likely to be at risk in subsequent years when encountering linear equations involving rational numbers.

Getenet and Callingham (2017) discussed the importance of fractions for students' future understanding of concepts such as proportional reasoning in deeper mathematical understanding and to support daily activities. They concluded that a student's competency with fraction concepts is a reflection of the use of language in the learning process and that the use of interrelated fraction concepts in conversation has implications for teachers' pedagogical and assessment approaches. Getenet and Callingham's study showed that Behr et al.'s (1983) model can be applied to normal classroom discourse. Students in their study were able to discuss fractions as part-whole relationships; they also used language consistent with ratio, operator, and quotient with teacher encouragement. Getenet and Callingham recommended developing the language use of the more unfamiliar terminology to make the sub-constructs of ratio, operator, quotient, and measure explicit.

2.3.6 Misconceptions and Error Patterns

Research shows that students tend to have misconceptions and often make errors when dealing with fractions. Frequently appearing errors are well known and have been comprehensively documented over many years (e.g., Ashlock, 1994; Carpenter, Fennema, & Romberg, 1993; Eichelmann, Narciss, Schnaubert, & Melis, 2012; Kerslake, 1986). The most cited common errors range from the most basic types such as the inability to arrange fractions in ascending or descending order; grouping errors, such as expressing a mixed fraction as $1\frac{26}{21}$; basic fact errors, such as $\frac{4}{10} = \frac{1}{2}$; and incorrect operations, such as $\frac{13}{15} + \frac{8}{15} = \frac{5}{15}$. Errors in more complex problems included the inability to apply fraction concepts in solving word problems, with errors ranging from comprehension errors to encoding and transformation errors (Yusof and Malone, 2003). Yusof and Malone (2003) concluded that student prior knowledge influenced misconceptions. They reported that a number of students confused fraction concepts with whole number concepts, and suggested a possible reason for this was because the students' prior learning of whole numbers had a negative influence on their understanding of fractions and their operations. This supported prior research by Post, Cramer, Behr, Lesh, and Harel (1993) and Moss and Case (1999), who also demonstrated that student's whole number schemes interfered with their efforts to learn fractions. Moss and Case found that when confronted with problems that required a new procedure to be utilised, students made mistakes that involved confusion of the rational numbers with whole numbers.

Wittmann (2013), who studied the difficulties students had with fraction computation, defined an "error pattern" as being evident when identically structured errors appear in the solution of two or more identically structured questions (e.g. $\frac{a}{b} \pm \frac{c}{d} = \frac{a \pm c}{b \pm d}$ for addition or subtraction of two fractions). If only some solutions show the error pattern and other solutions are correct, then the error pattern is not consistent. A student would need to deal with two or more identically structured problems and produce the same error for each for the error to be considered consistent. The consistency of error patterns in computational problems with fractions is still unknown.

Early studies (e.g. Hart, 1981; Padberg, 1986) identified frequently occurring "typical" errors of an entire test population, with Padberg also identifying error patterns consistently occurring at an individual level ("systematic errors"). Padberg did not, however, provide any data answering the question if error patterns are consistent for a given individual. In an empirical study focusing on the consistency of students' error patterns in solving computational problems with fractions, Wittmann (2013) reported that errors did not result from rationally chosen solutions and approaches were not consistent. The research implied that, generally, students did not have strategies when dealing with computational problems, instead their solutions were emergent while treating the problem. Wittmann, therefore, concluded that the consistency of an error pattern must be investigated at an individual level.

2.4 Diagnosis and Traditional Intervention

Understanding the patterns of reasoning behind students' answers in a fraction assessment might be a powerful aid for the design of pedagogical interventions. Understanding common mistakes also allows the provision of corrective feedback (Gómez, Jiménez, Bobadilla, Reyes, & Dartnell, 2014). In a study that analysed fifth grade students' misconceptions and error patterns when working with fractions, Morales (2014) concentrated on equivalence, addition, and subtraction. She suggested that by analysing how students use their knowledge — either conceptual or procedural, or a combination of both — instruction can be improved to meet the needs of the students who exhibit erroneous patterns in computation. Voza (2011) also found that identifying the types of errors students make when working on mathematics problems is important for teachers in that they can use appropriate activities to reteach the concept. Voza also suggested that when teachers do not recognise the error patterns students have, this can lead to an unmotivated and discouraging environment for students.

Morales (2014) concluded that students used both conceptual and procedural understanding when working with equivalence, addition, and subtraction of fractions. The students in Morales' study used pictures, gave examples, and made connections to other maths concepts and to daily life topics that showed they had conceptualised somehow what a fraction is. When working with addition and subtraction they reverted to their conceptualisation of fractions and used pictures to offer explanations for what they did.

Wittman (2009) investigated students' work with fraction computation, including multiplication of two fractions, addition and subtraction of two fractions, and addition of a fraction and an integer. Wittman's study highlighted that the numbers given in computational problems with fractions, especially denominators, have an effect on (1) whether students work on the problem or skip it, and (2) what kinds of approaches occur. The number of correct responses decreased as the denominators got larger. Wittman was surprised by the fact that the given numbers had an effect on the approach and hypothesised that the avoidance related to how well the students knew whole number multiplication and whether these could be calculated mentally without difficulty.

Past research has focused on diagnosing misconceptions in mathematics through individual interviews or by classifying errors on large scale assessments associated with particular misconceptions (Resnick, 1989; Vamvakoussi & Vosniadou, 2010). According to Durkin and Rittle-Johnson (2014) interviews provide a means for students to justify their answers or explain their thinking while solving problems. Interviews have been useful for extensively investigating individuals' misconceptions, however, they are time consuming, difficult to implement on a large scale, and not a practical option for many classroom practitioners. Although pen and paper assessments are easier to implement on a larger scale, they do not allow instructors to easily distinguish between strongly held and weakly held misconceptions.

Durkin and Rittle-Johnson (2014) investigated the use of a three-measure instrument designed to assess students' knowledge in the domain of decimal fractions. The measurement instrument in their study included three measures to diagnose common misconceptions. The *misconception error* measure categorised students' errors as particular misconceptions based on their response patterns. The *confidence ratings* assessed how strongly students held misconceptions by asking them to rate how sure they were that they answered a subset of items correctly. The *general magnitude strategy* measure assessed the existence of misconceptions in the absence of other competing strategies, such as a whole number misconception. An important finding in Durkin and Rittle-Johnson's study was that during the intervention the students started to recognise that their conceptions were incorrect but did not necessarily understand the correct concepts and therefore used other, also incorrect, methods. They suggested that instructors observe the shift in misconceptions as this might reflect the importance of what students notice. Confidence ratings can also reveal whether misconceptions are really being addressed by students with misconceptions. Simply measuring correct knowledge may give false impressions that students have begun to address misconceptions. In their study, Durkin and Rittle-Johnson found that correctness improved primarily from a reduction in low confidence errors, as opposed to high confidence errors which did not change. The stronger the belief in the misconception, the higher the chance of error.

Shin and Bryant (2015) synthesised the results from some intervention studies focusing on instruction to improve fraction skills. Acknowledging the difficulty students encounter in understanding the various concepts of fractions, Shin and Bryant concluded that teachers must be able to access interventions that focus on critical concepts and procedures associated with teaching fractions. Leading professional groups around the world have identified important conceptual understanding and procedural knowledge for fraction instruction that students must master in preparation for algebra. In the United States of America, the National Council of Teachers of Mathematics (NCTM, 2000) offered recommendations for instruction on fractions in its Content Standards. In 2006 the NCTM published the *Curriculum Focal Points for Prekindergarten Through Grade 8 Mathematics*, which emphasised an understanding of fractions concepts and skills in Grade 3 through 6. In Australia, the *Australian Curriculum: Mathematics* focuses on fractions between Year 4 and Year 7 with content based on the *General Capability: Numeracy*. Using fractions, decimals, percentages, ratios and rates is one of the *organising elements* and is described as:

> … students developing an understanding of the meaning of fractions and decimals, their representations as ratios, rates and percentages, and how they can be applied in real-life situations. Students visualise, order and describe shapes and objects using their proportions and the relationships of ratios, rates and percentages to solve problems in authentic contexts. In developing and acting with numeracy, students:
>
> - Interpret proportional reasoning

- Apply proportional reasoning (ACARA, 2017).

As Shin and Bryant (2015) highlighted, equally as important as frameworks for instructions, are evidence-based instructional components for teaching fractions to students who struggle with mathematical understanding of fractions. Despite the improvements in frameworks and increased grade-level expectations, a high number of students still exhibit misconceptions of fractions due to a lack of conceptual and procedural understanding. Empson and Levi (2011) suggested that the connection of visual representations and fractions and helps students engage in mathematical reasoning of problem solving. The researchers recommended that mathematical tasks include number lines, area models and fraction-bar models. Shin and Bryant came to the following conclusions:

(1) There needs to be sufficient time for practice during fraction interventions, highlighting a connection between students' instructional time and opportunities to practice to instructional effects.

(2) The fraction intervention programs must link to the standards for mathematical content. In their analysis of 17 fraction intervention programs, 12 included addition and subtraction with like and unlike denominators and only eight targeted multiplications. Shin and Bryant emphasised the importance of strengthening the understanding of the inverse relationships between multiplication and division of fractions, including partitioning concepts. None of the studies included negative fractions and none addressed how to

represent fractions on a number line by linking conceptual and procedural knowledge.

(3) The most commonly used instructional component was concrete and visual representations.

According to Siemon (2006), dialogue in mathematics education was focused on the importance of fostering students' mathematical understanding. This led to a commitment to generate new learning goals for students less in favour of skills and facts, and more focused on student thinking. The rationale given for mathematics in the *Australian Curriculum: Mathematics* (ACARA, 2017) reflects this goal:

> The curriculum focuses on developing increasingly sophisticated and refined mathematical understanding, fluency, reasoning, and problem-solving skills. These proficiencies enable students to respond to familiar and unfamiliar situations by employing mathematical strategies to make informed decisions and solve problems efficiently.

Fazio and Siegler (2010) suggested that many students' fraction arithmetic reflects common misconceptions. They focused on three of the common misconceptions: treating fractions' numerators and denominators as separate whole numbers, leaving the like denominators unchanged in fraction multiplication problems, and misunderstanding mixed numbers. Fazio and Siegler suggested that teachers should lead group discussions about different computational procedures, and why some lead to correct answers while others do not. They concluded that students will gain greater conceptual understanding of fraction arithmetic when they understand why the procedures from whole numbers do not work, rather than just learning a new procedure for fractions.

2.5 Summary of Key Points

Research into students' mathematical errors has significantly influenced the areas of mathematics assessment and intervention. Students routinely regard mistakes as indicators of their own low ability, however mistakes in mathematics are important and students and teachers should value mistakes and move from viewing them as learning failures to viewing them as learning achievements. Students need to work on challenging work that results in mistakes, and their mistakes should be valued for the opportunities they provide for learning.

During the primary years, students often acquire individual views and dispositions toward the learning of mathematics that can last for the rest of their lives. Spangler (2011) suggested that dispositions such as curiosity and perseverance are personal habits that play a key role in the future success of mathematics. Students who enjoy mathematics and perceive its relevance have higher proficiency than students with more negative perspectives. Spangler reported that students become less positive about mathematics as they proceed through school; both in confidence and enjoyment. An implication for this is that mathematics learning should not only enable students to gain proficiency in skills and understanding but also promote the desire to use what has been learned. Positive learning experiences should be based on giving students the opportunity to think and reflect about their work. An understanding about a student's self-belief is crucial in providing these positive learning experiences.

2.6 Self-Efficacy

Teachers must report on mathematics achievement, but often know little about their students' self-beliefs. Mathematics self-beliefs have an impact on learning and performance on several levels: cognitive, motivational, affective, and decision-making (Bandura, 1997). Although self-motivation and self-belief are imperative to the learning process, unlike curriculum content there is no requirement for teachers to report on self-efficacy and therefore it is unlikely to be monitored. Assessment instruments tend to focus on mathematics content, rather than students' beliefs about their ability (Bonne, 2012).

Mathematics has received special attention in self-efficacy research due to its valued place in the academic curriculum and because it is used widely in measures of achievement. Findings suggest that mathematics self-efficacy is a predictor of academic achievement across a range of education contexts (e.g. Chen, 2003; Pajares & Graham, 1999; Pajares & Miller, 1994; Schunk, 1981; Schunk & Hanson, 1985). Researchers have been able to demonstrate that self-efficacy beliefs predict students' mathematical performances. Pajares and Kranzler (1995) found that the influence of self-efficacy on mathematics performance was as strong as was the influence of general cognitive ability. Despite a lack of literature supporting this finding for mathematics specifically, this notion was supported by Bandura (1997, p. 216) who stated, “perceived self-efficacy is a better predictor of intellectual performance than skills alone”.

2.6.1 History and Context

Bandura (1977) proposed *social cognitive theory* when he realised his own *social learning theory* was missing a key element, self-belief. In making this change he advanced the view of human functioning as the product of personal, behavioural, and environmental influences. In altering the label, Bandura emphasised that cognition plays a critical role in people’s capability to construct reality, self-regulate, encode information, and perform behaviours. How individuals interpret the results of their performance attainments informs and alters their environments and their self-beliefs, which, in turn, inform and alter their subsequent performances. This is the foundation of Bandura’s (1978, 1986) conception of *reciprocal determinism* (see Figure 2.4).

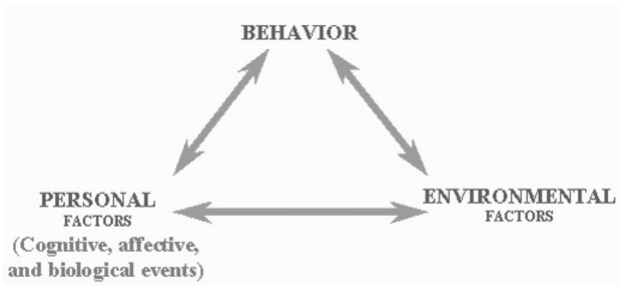


Figure 2.4. Bandura’s (1977, 1986) Model of Reciprocal Determinism.

Bandura (1986) considered self-reflection the most uniquely human capability, allowing people to evaluate and alter their own thinking and behaviour. These self-evaluations include perceptions of *self-efficacy*, “beliefs in one’s capabilities to organise and execute the courses of action required to manage prospective situations" (Bandura, 1995, p.2). According to Pajares (1996) these beliefs of personal competence affect behaviour in several ways. They influence the choices individuals make and the courses of action they pursue. People engage in tasks in which they feel competent and confident and avoid those in which they do not. Efficacy beliefs help determine how much effort people will expend on an activity, how long they will persevere when confronting obstacles, and how resilient they will prove in the face of adverse situations. The higher the sense of efficacy, the greater the effort, persistence and resilience (Pajares, 1996).

Bandura (1997) summarised that in order to realise their dreams, people try to exercise control over the events that affect their lives. He suggested that people have a stronger incentive to act if they believe that control is possible and that their actions will be effective. Bandura proposed that perceived self-efficacy regulates functioning in four ways: through cognitive, motivational, affective, and decisional processes. The effects of self-efficacy beliefs on cognitive processes take a variety of forms but consistently research has found that people with high self-efficacy are more likely to have high aspirations, think soundly, set themselves difficult challenges, and commit themselves to meeting those challenges. Bandura suggested people guide their actions by visualising successful outcomes instead of focusing on deficiencies or ways in which things could go wrong.

Self-beliefs of efficacy have been found to play a key role in the self-regulation of motivation. People motivate themselves by forming beliefs about what they can do, anticipating likely outcomes, setting goals, and planning courses of action designed to realise valued futures.

When addressing affective processes, people's beliefs in their coping capabilities affect how much stress and depression they experience in threatening or difficult situations, as well as their level of motivation. Efficacy beliefs regulate emotional states in a number of ways. People who believe they can manage threats are less distracted by them; those who lack self-efficacy are more likely to magnify risks. People with high self-efficacy lower their stress and anxiety by acting in ways that make the environment less threatening. Bandura believes that people are partly a product of their environment, therefore beliefs of personal efficacy can shape the course lives take by influencing the types of activities and environments people choose. Bandura (1994) firmly believed that people avoid activities and situations they believe exceed their coping capabilities, but they readily undertake challenging activities and select situations they judge themselves capable of handling. Bandura suggested this has a major influence over the person's life course, as people cultivate different competencies, interests, and social networks by the choices they make. The higher the level a person's perceived self-efficacy, the wider the range of career options they seriously consider, the greater their interest in them, and the better they prepare themselves educationally for future success.

During the 1970s research on human motivation was largely discussed in terms of "outcome expectations". Educators have long recognised that student beliefs about their capabilities play an essential role in their motivation. Bandura's most important effort to assess self-belief was when he focused on self-efficacy. It was during treatment of phobic individuals that Bandura first found individual differences in self-perceived capabilities.

Bandura labelled this individual difference *self-efficacy* and sought to measure it using task-specific scales. Although self-efficacy and outcome expectations were both hypothesised to affect motivation, he suggested that self-efficacy would have more influence anticipated outcomes were dependent on a person's judgment of how well they would be able to perform in given situations (Zimmerman, 2000). According to Bandura's (1986) *Social Cognitive Theory* what we do in any given circumstance is governed largely by what we think we can do.

Educational practices should be gauged not only by the skills and knowledge they impart for present use but also by what they do to students' belief about their capabilities, which affects how they approach the future. Students who develop a strong sense of self-efficacy are well equipped to educate themselves when they have to rely on their initiative (Bandura, 1986).

2.6.2 The Relationship Between Self-Efficacy and Academic Performance

Cognitive engagement and academic performance is enhanced if students have both knowledge and belief in their capabilities. In a study that examined relationships between motivational orientation and academic performance, Pintrich and DeGroot (1990) found that self-efficacy was positively related to student cognitive engagement and performance. Students who believed they were capable were more likely to report use of cognitive strategies, and to persist more often at difficult or uninteresting academic tasks. Pintrich and DeGroot concluded that self-efficacy played a mediational role in relation to cognitive strategies, and thereby, higher performance, and that "students need to have both the 'will' and the 'skill' to be successful in classrooms" (p. 38). Graham and Weiner (1996) reviewed the history of the study of motivation and the more specific principles that pertain to academic striving. They found that high self-efficacy and improved performance result when children adopt short-term goals, are taught to use specific learning strategies, and receive performance-contingent rewards. Graham and Weiner concluded that these instructional manipulations increase the belief that "I can do it", which, in turn, increases both effort and achievement. They believed that efficacy beliefs are related to the acquisition of new skills and to the performance of previously learned skills at a level of specificity not found in any other motivation conceptions.

Student motivation is something that affects all teachers. As Linnenbrink and Pintrich (2003) stated, teachers at all levels are always concerned with increasing student engagement and learning. Educators globally face the chronic problem of having some students involved, engaged and motivated, while others are disengaged and apathetic, even in the same classroom. Linnenbrink and Pintrich discussed self-efficacy in terms of how it may facilitate behavioural, cognitive, and motivational engagement in the classroom and provided specific practical suggestions for teachers. Behavioural engagement involves observable behaviour of the student in terms of their effort, persistence, and help-seeking. Do the students work hard, or are they distracted or putting in minimal effort? Do they persist at a task if they encounter difficulties and do they seek help when needed? Help-seeking in order to learn and understand is a good indicator of behavioural engagement, as opposed to executive help-seeking, where the student seek help simply in order to complete the task, or avoid doing any work. Pintrich and Schunk (1996) found that students who exert more effort, persist longer, and seek instrumental help are more likely to learn more and achieve at higher levels. When students are taught how to attribute improvement to their developing knowledge, skill, and effort, and are discouraged from attributing their poor performance to a lack of ability, this increases their self-efficacy.

Self-efficacy is closely related to learned helplessness. In self-efficacy theory *low outcome expectation* describes students who are considered 'learned helpless' when they have a belief that there is no relationship between their behaviour and the outcomes of learning and achievement. Linnenbrink and Pintrich (2003) believed that learned helplessness is often an issue for students with learning problems, or those in special education programs as they often have a long history of failure and attribute it to lack of ability. They see ability as uncontrollable and static.

Self-efficacy theory also states that simple behavioural engagement must be accompanied by cognitive engagement. Attention needs to be "hands on" and "minds on". Students need to be focused and thinking deeply about the content to be learned and need to think critically and creatively about the material. As much of student cognition takes place in the students' heads, it is difficult for a teacher to ascertain the level of cognitive engagement. Teachers can ask questions and listen to students' language to gauge cognition, but it may only provide a window on actual cognitive engagement. Pintrich and Schrauben (1992) have linked deeper processing strategies to both efficacy and achievement. Students who try to paraphrase or summarise the material or organise it in some way often display deeper, more conceptual learning.

The quality of cognitive engagement reflects the quality of students' effort in the task, whereas quantity of effort reflects behavioural engagement (Linnenbrink & Pintrich, 2003).

Teachers also want students to be engaged in the classroom tasks in terms of their interest, value, and affect. Students need to think that the content is important and worthwhile to learn, and teachers want them to have positive emotional or affective experiences while they are learning. These aspects are considered part of a students' motivational engagement. Eccles, Wigfield, and Schiefele (1998) demonstrated that students are capable of separating out their personal interest in a task from their values for a task, in terms of how important it is to them. Linnenbrink and Pintrich (2003) suggested there are three aspects of motivational engagement. Personal interest reflects the students' intrinsic interest, utility value represents how useful the students believe the content or task to be, and value beliefs about the general importance of the content or task in given their general goals in life. All three aspects of motivational engagement can be related learning and achievement.

Linnenbrink and Pintrich (2003) suggested that all three components of engagement are correlated. That is, for example, if students are cognitively and motivationally engaged, they are likely to be behaviourally engaged. After reviewing the research, they highlighted the important role efficacy plays in the way in which students engage and their achievement in school. They offered the following practical ways in which teachers can use efficacy to enhance engagement.

1. Help students maintain relatively high but accurate self-efficacy beliefs;
2. Provide students with challenging academic tasks that most students can reach with effort;
3. Foster the belief that competence or ability is a changeable, controllable aspect of development;
4. Promote students' domain specific self-efficacy beliefs rather than global self-esteem. (Linnenbrink & Pintrich, 2003, pp. 134-135)

Linnenbrink and Pintrich recommend, in the academic domain, that it is more important for students to have accurate feedback about their performance on a specific task, such as fraction knowledge, than be provided with general praise to develop self-esteem. The causal influence of self-efficacy on students' academic achievement-related behaviours has been effectively demonstrated by Schunk. In a progression of studies in the 1980s (e.g. Schunk, 1982a; 1982b; 1983a; 1983b; 1984a; 1984b; 1985), Schunk increased students' self-efficacy beliefs by providing them with instructional strategies designed to enhance their competence.

According to Bandura's (1986) social cognitive theory, students' self-efficacy beliefs mediate the influence of other determinants of academic outcomes on subsequent actions. Efficacy beliefs also act in unison with other common mechanisms of personal agency in influencing and predicting academic outcomes, and mathematics holds a valued place in the academic curriculum and has received special attention in self-efficacy research. Mathematical performance is easier to quantify than other areas of education and it also provides an important assessment tool for a broad range of educational purposes. Williams (1994) found the correlation between academic performance and self-efficacy higher in mathematics than any other domain.

Miller, Greene, Montalvo, Ravindran, and Nichols (1996) also reported that the degree of variation in students' self-efficacy beliefs is higher in the mathematics domain than any other academic area.

2.6.3 Mathematics Self-Efficacy

As suggested, most of the self-efficacy research in education has been in mathematics. Bandura (1997) defined mathematics self-efficacy as an individual's beliefs or perceptions with respect to his or her abilities in mathematics. An individual's mathematics self-efficacy is a student's confidence in completing a variety of tasks, from understanding concepts to solving problems. Generally, self-efficacy has been linked with motivation and many studies have established that students with higher levels of self-efficacy tend to be more motivated to learn than their peers and more likely to persist when presented with challenges (Pajares & Graham, 1999; Pajares & Kranzler, 1995; Zeldin, Britner, & Pajares, 2008). Although there is inconclusive evidence about the development of self-efficacy, researchers have consistently confirmed Bandura's (1997) four main sources of self-efficacy: mastery experiences, vicarious experiences, social persuasion, and physiological states (Hampton & Mason, 2003; Usher & Pajares, 2009).

According to May (2009) students base most of their beliefs about their abilities on their mastery of experiences. If a student succeeds in previous topics they believe they have the ability to succeed in future topics. In a study on designing a scale to explore the sources of mathematics self-efficacy, Usher and Pajares (2009) found that perceived mastery experience was a powerful source of students' mathematics self-efficacy. Students who felt they had mastered skills and succeeded at challenging assignments experienced an improvement in their self-efficacy. Vicarious experiences, in contrast, involve students observing social models similar to themselves succeeding with particular tasks. Although this does not contribute as strongly to self-efficacy as mastery experiences, student will feel more confident in mathematics if they see students they perceive as similar to themselves succeeding in mathematics. The two final sources contribute the least to students' self-efficacy. Social persuasion refers to encouragement, both positive and negative, from peers, teachers, and parents. Physiological states refer to the student's physical state such as fatigue, pain, or sense of well-being (May, 2009).

As reported by Linnenbrink and Pintich (2003), self-efficacy is different from self-concept, which reflects more general beliefs about competence (e.g., "I am good at mathematics"). Self-efficacy beliefs refer to much more specific and situational judgments of capabilities. For example, a self-efficacy judgement in mathematics might be expressed as, "I am confident I can solve these fraction problems". Self-efficacy theory proposes that these more specific judgments will be more closely related to an individual's actual engagement and learning than general self-concept measures. Pajares and Miller (1994) provided empirical support for this assumption, with research supporting the predictive power of self-efficacy over self-concept. A second aspect, according to Linnenbrink and Pintrich (2003), which distinguishes self-efficacy from self-concept and self-perceptions of competence, is that it is used in reference to some type of goal. This, again, reflects the more situational perspective of efficacy theory. The individual, task, or environment may determine the goal, but the judgments of efficacy are in reference to this goal.

In summarising their study of the role of self-efficacy beliefs in student engagement, Linnenbrink and Pintrich (2003) stated that self-efficacy plays an important role in student engagement in the classroom. Students who had positive and relatively high self-efficacy beliefs would be more likely to be engaged in the classroom in terms of their behaviour, cognition and motivation. This supports the findings of Pintrich and De Groot (1990) who found that junior high school students high in efficacy were more likely to report using various cognitive and self-regulatory or metacognitive learning strategies.

Self-efficacy is a motivational construct that is related to other motivational constructs, including personal interest and values such as utility and importance beliefs. There is a great deal of debate about the relative causal ordering of self-efficacy and other motivational constructs, such as personal interest and affect. For example, some researchers argue that students first like a task or topic and are then drawn to the activity due to their personal interest in the topic. As they engage with the activity over time, students develop expertise, knowledge, and skills, and from the development of expertise their self-efficacy beliefs develop (Renninger, Hidi, & Krapp, 1992). The interest-first perspective is a strong belief held by many teachers and they often worry how to interest students in content, and they see interest as a prerequisite to all learning and future motivation. In contrast to this belief, however, Eccles et al. (1998) found there was an alternative path to motivation and learning and suggested students' interest and value beliefs might develop out of judgments of competence. Bandura (1997) also suggested the same potential relation; that is, first individuals develop a sense of competence or efficacy at an activity, and from this they develop both interest and value for the activity. Regardless of primacy, it is more important to focus on the reciprocal relationship of self-efficacy and motivation. Linnenbrink and Pintrich (2003) suggested self-efficacy is more situational and open to change by contextual features than personal interest. Accordingly, a self-efficacy perspective suggests that if students are given challenging but achievable tasks, their efficacy will increase as they experience success.

Dweck (2006) showed that when students develop what she has called a "growth mindset" then they believe that knowledge can be learned and that the capacity of the brain can increase from exercise. The implications of this mindset are profound – students with a growth mindset work and learn more effectively, displaying a desire for challenge and resilience in the face of failure. Using Dweck's findings, Boaler (2016) suggested that mistakes are important opportunities for learning and growth, but students routinely regard mistakes as indicators of their own low ability. Using the findings from Dweck's work, Boaler proposed that every time a student makes a mistake in mathematics, new synapses are formed in their brain. When students have the opportunity to think about why something is wrong, new synaptic connections are sparked that cause the brain to grow in capacity.

Based on these perspectives, students who are provided with a method of solving fraction computation problems (such as via the old way/new way method) may feel more confident, and more motivated, in their mathematics lessons and may report higher levels of self-efficacy. Mathematics self-efficacy has also been associated with student's mathematics achievement. Lower achieving students are less likely to have previous successful mathematics experiences than higher achieving students and are therefore less likely to have higher levels of mathematics self-efficacy (Hall & Ponton, 2002).

Mathematics anxiety can also affect a students' motivation to learn in mathematics classes. Mathematics anxiety can be defined as the feelings of tension and anxiety that interfere with the manipulation of numbers and the solving of mathematical problems (Tobias, 1993). Cates and Rhymer (2003) found that students with higher levels of mathematics anxiety had significantly lower computational fluency in all areas of mathematical computations. Cates and Rhymer suggested that this lower level of fluency in turn decreases students' achievements in mathematics and likely contributes to the negative attitudes toward mathematics. Hembree (1990) hypothesised that students with low achievement in mathematics would develop negative emotions and attitudes toward mathematics, causing them to avoid mathematics in the future. This avoidance would cause students to continue performing poorly, confirming the students' emotions and attitudes toward mathematics. Once trapped in a vicious cycle, it would be difficult for students to alleviate their mathematics anxiety without some type of intervention.

Some aspects of mathematics appear to be cognitively difficult for many people to acquire; and some people have moderate or severe specific mathematical learning disabilities. According to Dowker, Sarkar, and Looi (2016), not all mathematical disabilities result from cognitive difficulties. After a far-reaching and in-depth review of the literature, they concluded that a substantial number of children and adults have mathematics anxiety, which may severely disrupt their mathematical learning and performance, both by causing avoidance of mathematical activities and by overloading and disrupting working memory during mathematical tasks.

2.6.4 Assessing Self-Efficacy

Pajares (1996a) emphasised that for a measure of self-efficacy to be reliable, it should require students to make judgments about their ability to solve specific problems. Pajares stressed that in order to accurately predict academic outcomes from students' self-efficacy beliefs, "self-efficacy judgments should be consistent with and tailored to the domain of functioning and/or task under investigation" (Pajares, 1996a, p. 547). This supported the notion from Bandura (1986) that self-efficacy beliefs should be assessed at the optimal level of specificity that corresponds to the critical task being assessed and the domain of functioning being analysed.

2.6.5 Summary

Using multiple theories in mathematics education research to explain the complexities of learning can provide a better explanation than a single theory might (Cobb, 2007).

Context-specific self-efficacy beliefs have been found to have greater predictive power for future achievement than do combined self-concept beliefs, and, further still, Bandura (1986) suggested that task-specific self-efficacy beliefs are even more accurate predictors. Bandura argued that reasonably precise judgments of capability matched to a specific outcome afford the greatest prediction and offer the best explanations of behavioural outcomes.

Based on the research presented, a mathematics self-efficacy scale, modified to focus on fractions, was used in this study. Details of the questionnaire can be found in section 4.5.5.

Chapter 3

Literature Review: Proactive Inhibition and the Old Way / New Way Strategy

3.1 Introduction

Misconceptions arise from the active role students take in constructing their own understanding. In mathematics, students regularly develop their own computational algorithms, often containing errors that are resistant to conventional remediation. The failure of conventional remediation methods to have long-term benefits and the effects of learning on change can often be related to transfer (Cole & Chan, 1990). Transfer can be described as the influence of one learning task on the learning and recall of another. Facilitative factors in recall are known as positive transfer and inhibitory factors are known as negative transfer. During the transfer phase students' construction of knowledge is sometimes incomplete. Students may develop some understanding of new concepts but when the concepts are interrelated they often have competing information and the competing information prevents transfer. Positive transfer is crucial to the success of remediation.

It is proposed in this study that a specific brain mechanism is responsible for the difficulties in transfer associated with traditional remediation. Lyndon (1982) found that proactive inhibition was principally responsible for preventing transfer under the conditions found in the use of traditional remediation. Lyndon devised a remediation strategy to overcome the effects of proactive inhibition. The Old Way/New Way strategy is a remedial technique applicable to situations that require a change in what has been learnt. The O/N strategy facilitates the change from repeated errors to positive transfer. Use of this strategy has been shown to have positive results for conceptual development, skill acquisition and self-esteem.

3.2 Errors

Humans are fallible and errors are expected. The causes and implications of human error have been studied extensively in the areas of health, industry, aviation, sport, and education (Reason, 1990). Radatz (1980) suggested errors are the product of previous experience; are causally determined, and often systematic; are persistent until there is intervention; can be analysed and described as error techniques; and can be derived from certain difficulties experienced while receiving and processing information. Error analysis develops a picture of the type of difficulty the learner is experiencing. Neuroscientists believe that the brain consists of highly orchestrated sets of fundamental building blocks termed "computational primitives" that are used for constructing sequences, retrieving information from memory, and routing information between different locations in the brain (Marcus, 2015). One such primitive, the behaviour sequence recording primitive, involves groups of neurons that correspond with sequences of behaviours that are often utilised in the same order. Coward (2014) demonstrated that sequences and combinations of primitives implement the different major types of memory observed in humans and a wide range of cognitive tasks. According to Radatz (1980), the ways in which errors manifest themselves are inextricably associated with these computational primitives, in which stored knowledge structures are selected and retrieved in response to situational demands.

Errors occur in various contexts, which involve multiple causes and lead to different learning potentials. The same learning methods that lead to knowledge under some circumstances also lead to error under other circumstances. As Reason (2003) described, correct performance and systematic errors are two sides of the same coin. The learning of new knowledge quite often arises from making errors with, and discovering the inadequacy of, our existing knowledge. According to McTavish and Larusson (2014), errors help frame "normal". McTavish and Larusson suggested that in the context of learning, the types of errors that are revealed in a task demonstrate areas of confusion and the hurdles that need to be overcome to attain mastery. Errors often correspond to one or more of the knowledge components, appearing in the skills, concepts, or the rules of a problem.

Our knowledge base renders us liable to confirmation bias. Confirmation bias is described as the tendency to search for, interpret, favour, and recall information in a way that confirms one's pre-existing beliefs. Confirmation bias occurs from the direct influence of desires on beliefs. Retrieval systems, capable of locating relevant items within a virtually unlimited knowledge base, lead our interpretations of the present and anticipations of the future to be shaped too much by the matching regularities of the past. Reason (1990) suggested that errors are much less common than correct actions as they are the failure of a planned action to achieve a desired outcome, they are considered a deviation from intention. Errors appear in very similar guises across a wide range of activities, and it is possible to identify comparable error forms in action, recognition, problem solving, decision-making, and concept formation. According to Reason (1990), error forms are recurrent varieties of fallibility which appear in a variety of cognitive activity, irrespective of error type.

3.2.1 The classification of errors

According to Reason (1990), there is no universally agreed classification of human error. Instead existing taxonomies reflect three levels at which classifications have been attempted: the behavioural, contextual, and conceptual levels. At the behavioural level, errors can be classified according to an easily observable feature of the erroneous behaviour. These features include either the formal characteristic of the error (e.g. repeating an action and omitting an action), or its more immediate consequence. Hollnagel (1993) classified erroneous human behaviour based on its observable manifestations as divergences from planned or normative sequences of actions.

Going beyond the formal error characteristic, the contextual level involves assumptions about causality. Errors at this level include reference to contextual triggering features such as perseverations. In psychology, perseveration is the repetition of a particular response, regardless of the absence or cessation of a stimulus. Cognitive perseveration is when one uses a previously used cognitive strategy inappropriately for a new or different task. For example, in mathematics a student may continue to divide by two, a strategy that has been successful, but not appropriate for a new question. Contextual categorisations of error are valuable as they draw attention to the interaction between triggering factors and the underlying error tendencies. Attention can be paid to what prompts an error to occur at a particular point in the behavioural sequence. In the example of dividing by two, the triggering factor might be confirmation bias, or a systematic error of reductive reasoning.

The conceptual level of error classification is based on assumptions about the cognitive mechanisms involved in error construction. According to Reason (1990), conceptual classifications are based more on theoretical inferences than the observable characteristics of the error. In mathematics, knowledge of procedures is not a guarantee of conceptual understanding; for example, a student may execute the correct procedure to divide fractions but, when confronted with a situation requiring fraction division, may misunderstand which number is the dividend and which is the divisor.

3.2.2 Repeated errors

Individuals do not bring to, or engage in, work tasks with a uniform base of experience or knowledge. Instead, they have diverse and personally distinct bases for conceptualising and construing what they experience. From engaging in particular sets of experiences in educational and other settings, they have learnt and continue to learn through ongoing and everyday problem-solving processes. Individuals either perform or fail to perform the requirements of a particular task or work. Through the process of encountering experiences, individuals develop a repertoire of knowledge which has conceptual, procedural, and/or dispositional dimensions. According to Billett (2012), the process of learning is comprised mostly of making mistakes with and discovering the inadequacy of our existing knowledge.

An incorrect action may be a result of a careless mistake or it may be an inevitable consequence of an antecedent event. The term "mistake" refers generally to errors due to carelessness, whereas a "misconception" (or repeated error) is a frequently observed misapplication of a concept or when someone has misleading ideas. Misconceptions are a result of patterns of error, or the systematic and observable procedures people use. Infrequently occurring mistakes rarely have a profound impact. Repeated errors, however, may be costly in terms of safety, health, money, and time (Reason, 2000). At times, errors may have major implications, may be unsafe and affect many people, and other times they may simply result in not achieving a personally desired outcome. Repeat errors may be obvious or go undetected. Errors can prevent or hinder skill development, affecting cumulative learning. Repeat errors have been demonstrated in a wide spectrum of human performance where automated skill, knowledge, or behavioural routines are involved.

In the context of mathematics learning, errors can be classified as either consistent or careless. According to Cox (1975), it is only when a student demonstrates an error pattern at least three times that indicates the error is habitual and automatic. In a study where students were asked to order decimal fractions, Resnick et al. (1989) were interested in the ways in which students used pre-existing knowledge to construct a mental representation of a new domain of knowledge. They found that student's errors were derived from their attempts to apply previously learned concepts, or notational conventions, to a new domain. Resnick et al. explained that, in a classroom, teachers provide various examples of mathematical procedures for student to learn and practise. When students are faced with a computation that they do not fully understand, students often decide for themselves how to proceed. Resnick et al. described these actions as the student's interpretation of the mathematical situation at the time; "errorful rules are a natural result of children's efforts to interpret what they are told and go beyond the cases actually presented, therefore errorful rules are active constructions" (p. 25).

Brown and Van Lehn's (1982) "Repair Theory" also described students' active construction of error patterns. Repair theory explains the process of how students develop consistent patterns of error when they are confronted with tasks that they are unsure of how to perform. Brown and Van Lehn hypothesised that students use a simple "repair" tactic to enable them to produce a solution. Brown and Van Lehn concluded that if the repair is left unchecked, the incorrect response, through habit, becomes a habit.

Having to change one's own established ways in the face of new and conflicting knowledge is the derivation of habit errors. Repeated errors are described as knowledge for those performing them, rather than the result of a lack of knowledge. For this reason, the wide prevalence of habitual error patterns has serious implications for corrective attempts (Reason, 1990).

3.3 Errors and the Interference Effect of Prior Learning

The interference effect of prior learning has been attributed to a psychological phenomenon known as *proactive inhibition*, also known as *proactive interference* (Baxter, 2000; Lyndon, 1989; Underwood, 1966). Underwood found that proactive inhibition (PI) caused accelerated forgetting of the new information the person was trying to learn. Proactive inhibition is an information protection mechanism: it protects all learned knowledge and skills – right and wrong – and strongly resists and slows down any attempts to change or improve prior knowledge.

Proactive inhibition also exerts significant control over the amount of information that can be retrieved from working memory. The study by Keppel and Underwood (1962) reported that forgetting from working memory is attributed to the interference caused by prior material, that is, old information stored in long-term memory. They found that information from long-term memory was interfering with new information, due to the similarity of the information presented. They concluded that these results were attributed to the occurrence of proactive interference.

A study by Jonides and Nee (2006) postulated that if working memory is critical to normal cognitive functioning, then proactive interference is an important determinant of the success of working memory. Jonides and Nee summarised that working memory is a critical capacity underlying many higher cognitive functions, and that performance of higher cognitive functions is closely related to the ability to resolve proactive interference from previous information. Concluding that resolving interference among items in memory is a critical cognitive skill that has important implications for a host of other cognitive skills, their study examined the brain mechanisms that resolve interference. Jonides and Nee's study looked specifically at the brain mechanisms of proactive interference in working memory and found that it was activated in the left lateral frontal cortex, including the same inferior frontal region investigated by Smith et al. (2001), suggesting a relationship between working memory capacity and resolution of proactive interference. Jonides and Nee (2006) postulated that reduced working memory capacity actually resulted from proactive interference due to retrieval competition. Retrieval competition is a phenomenon where item retrieval causes forgetting of other information. This supported earlier research conducted by Anderson and Neely (1996) who reported that the measured capacity of working memory depends on how many items can be retrieved, and proactive interference affects measured capacity by making retrieval more difficult. This is significant given the demonstrated centrality of working memory capacity as a predictor of other cognitive skills, such as mathematics learning in general and numeracy in particular.

As described by Munro (2011), successful mathematics learning makes unique demands on working memory processes. Working memory refers to a mental workspace, involved in controlling, regulating, and actively maintaining relevant information to accomplish complex cognitive tasks. Students engage their working memory when they interpret the information they are taught using knowledge they retrieve from long term memory, when they retain and link mathematics ideas to synthesise new mathematics knowledge, and when they direct their learning and thinking activity to compute or solve mathematical problems.

Proactive inhibition does not prevent learning from occurring; it merely prevents the association of conflicting ideas and protects all prior knowledge (Hanin, Malveda & Hanina, 2004). PI exhibits a maintenance effect over prior learning, inhibiting change and preserving erroneous (as well as correct) knowledge and skills. PI is an involuntary mechanism, however, and an individual's level of interference varies (Stroop, 1935). The level of PI an individual has is not associated with measures of intellectual ability (Baxter, 2000), however, individuals with higher PI are less likely to achieve successful behaviour change (error habit reversal) under conventional correction methods. Lyndon (1989) suggested remediation of erroneous errors therefore needed to embrace the influence of PI to be effective.

This level of understanding suggests that the effects of proactive inhibition and the order of acquisition of information have a significant bearing on the learning and retention of conflicting information. Remedial techniques traditionally rely on practice and repetition and pay little attention to prior knowledge and how it interacts with new knowledge.

3.4 Remediation of Errors

Conventionally, errors were thought to occur as a result of intellectual or perceptual deficits. Kephart (1960) described how, under the deficit model, errors are a sign that learning did not take place and suggested that consistent and persistent error implies a lack of knowledge or skill. The solution has been to re-teach or re-train, which can be time consuming, resource expensive, and largely unsuccessful (Connell & Peck, 1993; Dole, 1991; Read, 1987). Even when learning gains are made during conventional retraining, these improvements often fail to transfer to situations outside the original setting where the retraining took place. This happens because the cues for correct performance are withdrawn and short-term gains are not permanent. When put under pressure or faced with a stressful performance situation, reversion to old incorrect habits is commonly experienced. This is not indicative of a deficit in understanding; rather it is indicative of poor transfer of learning and becomes a "transfer problem".

Transfer problems pose an obstacle to the learning progress wherever automated skill, knowledge, and/or behavioural routines are involved. The concept of transfer derived its meaning from research into the transfer of learning, that is, the influence of one learning task on the learning and recall of another. According to Lyndon (2000), a broad conceptualisation of transfer referred to "that which was carried over from one experience to another" (p. 43). Research into transfer has been divided into the study of facilitative factors in recall, or "positive transfer", and inhibitory factors, or "negative transfer". When a change in concept or performance is required – a "response substitution" in behavioural terms – and the intervention is successful, it would be considered a positive transfer and a generalisation of change is likely. Lyndon (2000) suggested that if the intervention is unsuccessful, rather than the result being due to failure to generalise a new concept, the outcome is best described as being due to negative transfer. Lyndon postulated that negative transfer is due to the influence of prior learning on the learning and subsequent recollection of a conflicting response to an old stimulus. If a student's prior knowledge affects his or her ability to learn and to recall a conflicting, although correct, response, this demonstrates the effects of negative transfer and proactive inhibition.

According to Baxter, Lyndon, Dole, and Battistutta (2004), the significance of consistent and persistent errors, and misconceptions as obstacles to learning new ideas and learning new ways of doing things, is often underestimated. They suggested that habit pattern errors are among the most common of all error forms and the most difficult to eradicate. Lyndon (1989) asserted that once a habit has formed, it is difficult to adjust the behaviour because the old learning disables new learning by the process of psychological interference. He developed an innovative teaching method to deal with the interference effects of proactive inhibition. *Old Way/New Way* (O/N) method is a synthesis of existing concepts and principles, including automaticity in behaviour (Bargh & Chartrand, 1999), learned errors (Reason, 1990), the influence of prior learning (Ausubel, 1968), metacognition (Flavell, 1987) and proactive inhibition and accelerated forgetting (Underwood, 1957, 1966). The O/N strategy has produced successful results in various settings: skill correction for sporting techniques, aviation, workplace safety, science education, and spelling. It has also had positive results in work with subtraction algorithms (Dole, 1993) and percentages (Dole, 2003) in mathematics education.

The traditional approach to remediation faces the problem of negative transfer, or proactive inhibitory effects. When teachers attempt to teach correct responses to old stimuli, there is a known natural inhibitory factor that accounts for the lack of success of conventional remediation. The disruptive effects upon the recall of newly acquired associations are due to the fundamental and normal phenomenon of proactive inhibition. The powerful affective and interfering affects of proactive inhibition must be acknowledged and considered in attempts to understand the problems associated with conventional remediation. Lyndon (2000) listed three factors known to facilitate positive transfer: stimulus discrimination, response discrimination, and response practice. Stimulus discrimination is the response to certain stimuli, not responding to those that are similar. Response discrimination is when someone is wishing to learn a new but conflicting response to an old stimulus. Response practice involves, as the name suggests, practicing the new response. Lyndon argued that at the same time a student is confronted by the greatest potential for negative transfer and increasing proactive inhibition, they also have the highest potential for positive transfer. Lyndon proposed a method for incorporating positive transfer factors to effect a change in behaviour. The Old Way / New Way strategy differs from conventional remediation in the importance placed on the student's representation and elicitation of personal knowledge. In the O/N strategy the error represents what the student does know and focuses on the three positive transfer factors: stimulus discrimination, response discrimination, and response practice (Lyndon, 2000).

3.5 The Old Way / New Way Strategy

Lyndon (1989) developed an innovative teaching method to overcome the interference effects of proactive inhibition. Based on countering the interference effect of prior learning, the Old Way/New Way strategy (sometimes referred to as *Mediational Learning*) has successfully been demonstrated in a wide variety of applications where changes in habit, skills, and concepts are required. Lyndon (1989) suggested that emerging research in cognitive psychology indicates that learned habit patterns influence and direct our daily thoughts and actions, including performance in sport or at work; our conceptual framework, including misconceptions; our ability to learn; and how we interact with others. According to Lyndon, these learned behaviours – whether right or wrong, suitable or unsuitable, effective or ineffective, well-adjusted or maladjusted – are under the influence of habit forces. Lyndon suggested these habit patterns automatically develop during practice, and the better someone has practiced and therefore habituated the thought, performance, or behaviour, the harder it is to change. Old Way / New Way is reported to be a powerful, cost- and time-effective learning method that can change habits quickly and permanently.

The strategy involves bringing the learner's "old way" to a conscious level and exchanging it for a "new way" by means of discrimination learning. This is followed by practice with the correct "new way". The O/N methodology consists of protocol and prescribes a sequence of four steps that are followed in the correct sequence. The protocol is claimed to accelerate cognitive and behavioural change within individuals, greatly reduce the typically prolonged adaptation period to the adoption of change, and improve learning transfer (Lyndon, 2000).

3.5.1 Comparisons of O/N strategy to conventional approaches

O/N differs from conventional approaches to error correction as it bypasses the brain mechanisms that preserve prior learning. This difference is the key to its potential effectiveness. The two main differences are that persistent and consistent errors are considered a sign that learning has occurred, rather than a sign of learning failure; and, secondly, habitual errors must be acknowledged and identified and deconstructed, not ignored.

Dole (1999) found the O/N shared similarities with other procedures for dealing with errors/misconceptions (e.g., Borassi, 1994; Gable, Enright & Hendrickson, 1991; Rauff, 1994) but its method was more prescriptive and firmly based on psychological principles of learning. In Borassi's (1994) study students were asked to analyse, compare, and contrast fellow students' work, justifying or rejecting their responses. The role of the teacher was to assist the inquiry process, prompting students to explain clearly their statements and to probe their knowledge. According to Borassi, the strategy of using "errors as springboards for inquiry" increased the students' learning of mathematical content as a result of the teaching experiment. Borassi also reported an increase in the affective domain of the students, with students reporting that they felt more positive about the study of mathematics, and their own ability to continue with the study of mathematics. Borassi's "errors as springboards for inquiry" strategy was similar to the O/N strategy in that students compared incorrect definitions and contrasted them to their own understanding, engaging them in conceptual analysis of a concept. The strategy differed from O/N in that the errors the students were analysing were not their own.

Gable, Enright, and Hendrickson (1991) were more prescriptive than Borassi in their approach to the correction of error patterns in arithmetic. They described a three-phase model, with the first phase being the identification of the consistency of the error, which involved interviewing the student. The second phase begins the intervention, and involves three stages of demonstration of the correct algorithm, selection of "error groups and appropriate corrective strategy" (p. 7), and practice of the new algorithm. The appropriate corrective strategy is through categorising the nature of the error. Phase two is characterised by extensive practice of the new/correct computational procedure. Phase three is the evaluative phase, addressing the application of the skill in the regular classroom.

Rauff (1994) suggested that a process of "belief-based teaching" can help students overcome inappropriate mathematical procedures, and described this in terms of students' erroneous solutions for factoring polynomials. In a process similar to Borassi (1994), Rauff suggested that, to overcome students' misconceptions about particular mathematical procedures they must first be determined, and then the teacher's role is to assist the integration of the appropriate mathematical procedures within the student's belief set. Rauff summarised belief-based teaching thus:

> The focus of this approach into teaching and learning is student belief. An instructor using this approach to teaching factoring begins with asking the student to tell him or her what they think about factoring. The instructor then analyses their "buggy" factorisings in light of their beliefs. The students are next shown how their beliefs produce non-equivalent expressions. Finally, the students modify their beliefs appropriately. (p. 425)

The O/N strategy differs from the methods of Borassi (1994), Gable, Enright, and Henderson (1991), and Rauff (1994) as O/N is a prescriptive series of steps where student misconceptions are actively differentiated between by the students with the assistance of the teacher. The O/N strategy is a dialogue where the teacher engages in a mediating role between students' knowledge and the misconception.

Hanin et al. (2002) highlighted how the O/N strategy represented an individualised approach and that a promising dimension to the O/N methodology came from its connection with metacognition, which is an important aspect of modern learning theory in academic studies. According to Chick, Karis, and Kernahan (2009) metacognitive practices help students become aware of their strengths and weaknesses as learners, which in turn help the learner to de-automatise a learned error. It is important to realise that the core of the O/N learning trial is not merely increasing a learner's awareness of erroneous and correct methods of task execution. Rather, what matters most is the activation of a mediation process contrasting old and new patterns that is crucial for overcoming proactive inhibition (Dawson & Lyndon, 1997).

In an attempt to resolve the remediation problem of transfer in spelling errors, Lyndon (1989), reported higher rates of transfer observed than conventional methods. In a study involving 25 students, ranging from ages 7-12, Lyndon compared the O/N technique to the traditional "Look-Say-Cover-Write-Say-Check" remediation approach. He found that what a student pays attention to, either voluntarily or involuntarily, determines what is learned, and what the student knows prior to an experience will determine what is available for conscious recall. He also reported that only one trial was usually insufficient for full elimination of the old way due to the phenomenon of spontaneous recovery, and so suggested multiple trials. Spontaneous recovery of learning and memory refers to the re-emergence of the previously extinguished conditioned response after a delay (Underwood, 1966). Lyndon concluded that up to four or five trials spaced two weeks apart are required for positive transfer to take place.

Nicholls and Ward (1998) compared O/N to Vygotsky's 'internalisation' and Zone of Proximal Development. According to Vygotsky (1978), cognitive development results from an internalisation of language. Vygotsky suggested that dialogue between teacher and student, where the student is encouraged to talk about what they are able to do, helps the student to organise his/her intentions, thoughts, and actions. Sharing language leads first to speech and then to the development of inner speech and internalising. Vygotsky emphasised the importance of the teacher in promoting an interactive learning environment, enticing students into the learning process through social interaction and knowledge sharing.

Fox (1995) gave support to the O/N strategy when he claimed that discussion plays three roles in learning: it supports efforts to construct new meanings as they are explored in words, it allows us to test out and criticise claims and different points of views as we speak and listen to others, and it provides raw material for our own reflective thought.

Fisher, Bruce, and Greive (2007) compared O/N to a variation of a more traditional approach used widely in Australian schools, *Look-Say-Cover-Write-Say-Check,* as an effective spelling remediation approach. They utilised an experimental research design, as previous evidence of the effectiveness of O/N was anecdotal. Their results indicated that there was significant improvement in spelling scores from pre- to post-test, however, neither intervention was found to be more effective than the other. In this particular study, there was no follow-up testing to determine retention of either skill learnt.

3.5.2 Practical applications of the O/N strategy

Experimental and quasi-experimental studies and field trials in sport (Hanin et al., 2002), in mathematics and science education (Baxter & Dole, 1990; Baxter et al., 1999; Dawson & Lyndon, 1997; Dole, 1991, 1993, 1999; Henderson et al., 1999; Lyndon, 1989, 2000; Lyndon & Dawson, 1995; Rowell et al., 1990), in speech therapy (Lyndon & Malcolm, 1984), and in workplace training (Weaver et al., 2000) have consistently demonstrated support for Old Way/New Way. Lyndon (2000) maintained the O/N strategy has relevance in many areas of human learning where stable changes in habit, skills, and concepts are required. Regardless of the established habit, skill, or knowledge, if there is a need to learn new but conflicting information, the brain mechanisms involved are the same.

In a study about the correction of systematic errors, Baxter and Dole (1990) compared two different approaches to the correction of consistent subtraction errors. They used practical materials such as multi-base arithmetic blocks (MAB) and place value charts to model the subtraction algorithm and compared it to intervention involving the O/N strategy. Their study used six students, with a low achieving student paired with high achieving student (both with errors), and then randomly assigned to either of the two treatment groups or control group. In their study, the mean scores were compared and the results illustrated an improvement with O/N technique and no improvement in MAB blocks group. One of the control group members showed improvement but the reported mean gain was less than for the O/N group. Their study was not conclusive due to being on such a small scale and it only involved pre- and post-tests but no delayed retention testing.

Dole (1993) compared the use of the O/N strategy to a conventional method of instruction for correcting systematic computational errors and promoting subtraction knowledge growth in upper primary students. The conventional method was based on systematic and structured reteaching and focussed on linking symbolic subtraction procedure to pictorial representations. Dole was interested in comparing students' computational knowledge to that of the individual's intuitive, concrete, and so called principled/conceptual knowledge. Leinhardt (1988) suggested that knowing mathematics derives from those four knowledge types. *Intuitive* knowledge is the real-world application knowledge that is acquired before formal instruction, *concrete* knowledge is knowledge associated with representation by appropriate concrete materials during instruction, *computational* knowledge is the knowledge associated with formal procedures, and *principled/conceptual* knowledge is the underlying knowledge of appropriate concepts and knowing how to apply them in different contexts. In Dole's study, sixteen students were selected from a pool of 60 upon identification of systematic errors in subtraction via a diagnostic test. O/N was found to be successful in changing computational knowledge for all students, and in building concrete and principled/conceptual knowledge. The conventional approach was less successful in improving computational knowledge and marginally better in building concrete and principled/conceptual knowledge. Dole (1993) regarded O/N as a convergent remediation approach capable of melding the knowledge types holistically. She concluded that the superiority of the O/N method lay in the short amount of time and effort required for its implementation and its power to motivate students. She reported that the O/N method restored confidence in the individual's own ability to learn, and the structured and sequential activities appeared to be an effective means of promoting concrete, computational, and principled/conceptual knowledge.

Fitts and Posner (1967) proposed a model of motor skill acquisition that centred on three stages. In their theory, performance was characterised by three sequential stages, termed the cognitive, associative, and autonomous stages. In the cognitive stage, the goal is to develop an overall understanding of the skill. In the associative stage, the learner begins to demonstrate a more refined movement through practice. During the final stage of learning, the autonomous stage, the motor skill becomes mostly automatic. Correction of erroneous techniques in athletic performance is often ineffective because conventional skill correction methods rely mainly on re-teaching by repetition (Hanin, Malvela & Hanina, 2004). It is much less effective to change or improve an automated skill which is not under conscious control. As a practical way to address this problem, the O/N strategy has been applied to sporting performance by contrasting the erroneous and correct movement patterns by following an individually tailored protocol prepared prior to an intervention, called a "learning trial". Hanin et al. (2002) investigated a correction technique using O/N for Olympic athletes. They suggested that performance inhibiting learned errors interfere with skilled performance, and play cause and effect roles in sports injuries and postural problems of athletes. They proposed that when an error in technique goes uncorrected it progresses through to the autonomous stage of learning and is then harder to eradicate.

The impact of learned errors is that there is often a poor transfer of learning from skill drills to competitive performance. Like students transferring knowledge from the classroom to test situations, athletes often seem to improve during training and drills but fall back to their old incorrect ways in the heat of competition (Maschette, 1985; Young, 1985). Studies of proactive inhibitory effects on skill acquisition in sport are scarce, however, a study by Eason, Smith and Plaisance (1989) reported that a previously learned skill interfered with the learning of a new skill. They examined the effect of negative transfer theory of learning of the tennis forehand with the subsequent learning of the backhand stroke. The results, using an experimental and control group, indicated that learning the forehand independent of the backhand interfered with learning the backhand. The performance became cue-dependent and the individual reverted to prior behaviour patterns with the absence of those cues.

3.5.3 Practical application of a learning trial in mathematics

The four steps of the O/N strategy are reactivation, labelling, discrimination, and generalisation. To illustrate how the O/N method proceeds through the four steps, an example is provided showing the remediation of a systematic error in fraction addition. Prior to starting any trial, the individual's "error" is analysed. Reactivation of the error memory involves asking the individual to then perform the computation in their way. So, in step 1, *reactivation of the error memory*, the student is asked to complete the fraction addition problem $\frac{2}{3} + \frac{3}{5}$ in their usual way. For step 2, *labelling and offering an alternative*, the student is asked if that particular method of performing the computation can be called the "old way". When the student consents, the student is asked if a "new way" for computing $\frac{2}{3} + \frac{3}{5}$ can be demonstrated. The differences between the two computations are then pointed out usually carefully selected language. In step 3, *discrimination*, the student is asked to perform the computation the old way, then the new way, and then asked to contrast the two ways. This discrimination of the same problem ($\frac{2}{3} + \frac{3}{5}$) is repeated five times. In step 4, *generalisation*, the student is given six fraction addition questions and asked to complete them using the new way. The steps comprise one complete O/N trial, which takes approximately 10 minutes (Lyndon, 1989).

3.6 Summary

The purpose of this research is to further develop the knowledge of errors and error patterns in fraction computation and to determine the relative effectiveness of the Old Way/New Way strategy compared to a traditional intervention program for the remedial learning of fractions. Previous studies (Dole, 1991; Dole 2003; Yates & Lyndon, 2004) have reported the power of O/N to motivate learners, empower students with a way of dealing with misconceptions, increase confidence, and decrease school malaise. The effect O/N has on increasing confidence, and decreasing school malaise has only been reported but not quantified, therefore providing a future direction for research in this area.

Chapter 4

Research Design

4.1 Introduction

The purpose of this research is to further develop the knowledge of students' errors and error patterns in fraction computation and to determine the relative effectiveness of the Old Way/New Way strategy compared to a traditional intervention program for the remedial learning of fractions. Previous studies (Dole, 1991; Yates & Lyndon, 2004; Dole 2003) have reported the power of O/N to motivate learners, empower students with a way of dealing with misconceptions, increase confidence, and decrease school malaise. This effect has only been reported but not quantified, suggesting a future direction for research in this area. This chapter describes the design utilised in this research to achieve the aims and objectives stated in section 1.3.1 of Chapter 1, and specifically to address the three research questions:

- What are the difficulties encountered by high-school students when working with fractions?
- How does the O/N Technique compare to a traditional remediation program when addressing error computations in fractions?

- What are the effects of the O/N technique and traditional remediation on self-efficacy?

A pilot study was conducted in the year preceding the main study to trial the Fractions Diagnostic Test instrument. The participants for both the pilot and main study were from the same school. Section 4.2 discusses the methodology used in the study, the stages by which the methodology was implemented, and the research design; section 4.3 details the context of the study; section 4.4 lists the phases of the research; 4.5 describes the pilot study conducted and the final instruments chosen for the study; section 4.6 describes the main study; section 4.7 details the other data used in the study; section 4.8 describes the methods of data analysis; and, finally, section 4.9 discusses the ethical considerations of the research and its problems and limitations.

4.2 Methodology

Based on the three research questions, the problem-oriented pragmatism philosophy was adopted, allowing a mix of both quantitative and qualitative methods to be used. A pragmatic research philosophy embraces mix-method approaches to research questions. With its origins in the work of Dewey (1931), and contemporary support from Rorty (1990, 1991), pragmatism emphasises the practical problems experienced by people, the research questions posited, and the consequences of inquiry. The aim of philosophical underpinning to this research is to engage in dialogue where different types of knowledge are viewed as tools for helping us cope with and thrive within our environment (Rorty, 1990).

The methodology was intervention research using an experimental group design supported by qualitative components. According to Ross and Onwuegbuzie (2012) mathematics researchers over the last several decades have struggled to agree upon what represents the most appropriate research approach to use for research in mathematics education, leading to a form of research identity crisis. Lester (2005) recounted a call in the 1960s and 1970s to make mathematics education research more scientific, however, experimental research during that time was also criticised as being inappropriate for addressing questions of *what works.* Scandura (1967) concluded, "… people are critical of the quality of the research in mathematics education. They look at tables of statistical data and they say 'So what!' They feel that vital questions go unanswered while means, standard deviations, and t-tests pile up" (p. iii). Kilpatrick (2014) reported an abundance of research activity in mathematics education that has been integral part of its growth and development. The subject matter of mathematics education research has broadened to include school curriculum, assessment, the education of mathematics teachers, and professional development of teachers. Kilpatrick observed that the methods used now go well beyond experimentation to include case studies, surveys of attitudes and beliefs, and ethnographies of cultural practices.

As mathematics education researchers are yet to reach a consensus of an agreed standard, this research adopts the strengths inherent in mono-methods but combines both quantitative and qualitative research within the same inquiry. The intention of the mixed methods approach in this research is to provide the *why* answers and prevent the *so what.* As Creswell and

Garrett (2008) suggested, the researchers who gravitate towards mixed methods will also be providing the momentum needed for it to continue to be a 'movement' and educational sub fields, such as mathematics education, can place their individual stamp on the field.

Schoenfeld (2008) described mathematics education as well grounded in psychology and philosophy, and emphasised that, despite mathematics education having a long history, the discipline of mathematics education research was still relatively new. This explains why there is lack of a dominant research paradigm in mathematics education. In a research forum on the theories of mathematics education at the 29th Conference of the International Group for the Psychology of Mathematics Education (PME) English and Sriraman (2005) suggested a plausible explanation for the presence of multiple theories of mathematical learning is that "… mathematics education, unlike 'pure' disciplines in the sciences, is heavily influenced by cultural, social and political forces" (p. 171). Lerman (2000) suggested that the switch to research on the social dimensions of mathematical learning towards the end of the 1980s resulted in theories that emphasised a view of mathematics as a social product.

In considering the purpose of research in mathematics education, Schoenfeld (2000) proposed three questions: What kinds of questions can educational research answer? What kinds of evidence are appropriate to back up educational claims? What kinds of methods can generate such evidence? He described research in mathematics education as having two main purposes, one pure and one applied:

Pure: To understand the nature of mathematical thinking, teaching, and learning; and

Applied: To use such understandings to improve mathematics instruction.

Schoenfeld suggested it is often difficult to employ straightforward experimental or statistical research methods of the type used in physical sciences because of the complexities related to what it means for educational conditions to be replicable. The number and type of research methods in mathematics education have increased and studies employing anthropological observation techniques and other qualitative methods are increasingly common. Rather than focus on a particular method, Schoenfeld (2000) suggested a set of criteria for research methods in mathematics education: descriptive power, explanatory power, scope, predictive power, rigor and specificity, falsifiability, replicability, and multiple sources of evidence ("triangulation"). Whilst most of these criteria can be satisfied with experimental design, it is the multiple sources of evidence that sets social sciences apart from, say, mathematics. In mathematics, validity can be established through one proof. Schoenfeld suggested that education and the social sciences tend to look for *compelling evidence*. He suggested that evidence can be misleading, what we think is general may in fact be artefact or a function of circumstances rather than a general phenomenon. One way to check for artefactual behaviour is to seek as many sources of information as possible about the phenomenon and to see whether they portray a consistent message. In this way we look for convergence of data. Research in mathematics education is a very different enterprise from research in pure mathematics. Findings are rarely definitive; they are usually suggestive. A scientific approach is possible but scientific methods, such as the experimental method, should be supported by a variety of methods appropriate for the task (Ross & Onwuegbuzie, 2012).

Baker, Gersten, and Lee (2002) synthesised data from experimental studies that assessed the effects of interventions designed to improve the mathematics achievement of students considered to be low achieving. They provided a systematic observation of what had been learned about mathematics education through controlled research in classroom settings. They reported that in most cases very subtle aspects of curriculum design were manipulated in order to assess effectiveness. This supported the view of the National Research Council of the United States (Kilpatrick et al., 2001) that suggested, "Experimental rigor often requires narrowing one's focus to a single feature of an instructional method or to a limited amount of mathematical content" (p. 25). Baker, Gersten, and Lee (2002) suggested that focusing on student errors and misconceptions can also be an effective instructional method, especially when teachers anticipate predictable student errors and prepare in advance to use those errors to help students understand correct solutions. Baker, Gersten, and Lee also reported that principles of direct or explicit instruction could be useful in teaching mathematical concepts and procedures. This included both the use of strategies derived from cognitive psychology to develop generic problem-solving strategies and more classic direct instruction approaches where students are taught one way to solve a problem and are provided with practice. This was particularly successful with concepts involving fractions, ratios, and decimals. Kilpatrick et al. (2001) suggested "high-quality research should play a central role in any effort to improve mathematics learning. That research can never provide prescriptions, but it can be used to help guide skilled teachers in crafting methods that will work in their particular circumstances" (p. 26).

Most of this research is highly quantitative in nature, with the majority of the data being derived from paper and pencil diagnostic pre- and post-tests and pre- and post-self-efficacy questionnaires. However, this research also adapts some elements of qualitative research paradigm such as work samples, reflective journals and ad-hoc interviews.

4.2.1 Mixed Methods

Creswell and Plano Clark (2007) described mixed methods as an approach to inquiry in which the researcher links both quantitative and qualitative data to provide a unified understanding of a research problem. Johnson, Onwuegbuzie, and Turner (2007) offered the following definition of mixed methods research:

> Mixed methods research is an intellectual and practical synthesis based on qualitative and quantitative research … It recognises the importance of traditional quantitative and qualitative research but also offers a powerful third paradigm choice that often will provide the most informative, complete, balanced, and useful research results. (p. 129)

Collins, Onwuegbuzie, and Sutton (2012) identified four common rationales for mixing quantitative and qualitative research approaches: participant enrichment, instrument fidelity, treatment integrity, and significance enhancement. In an examination of the prevalence of mixed methods research in mathematics education, Ross and Onwuegbuzie (2012) suggested each of the four rationales could come before, during, and/or after the study. Participant enrichment optimises the sample, whereas instrument fidelity maximises the appropriateness and utility of the instruments used in the study. Treatment integrity pertains to the combining of quantitative and qualitative techniques for the rationale of assessing the fidelity of the treatment programs or interventions. The significance enhancement involves the use of qualitative and quantitative approaches to maximise the accuracy of the interpretation of the results. Using both quantitative and qualitative data analysis techniques either concurrently or sequentially within the same study can fulfil one or more of Greene, Caracelli, and Graham's (1989) five purposes for integrating quantitative and qualitative approaches: triangulation (comparing results from quantitative data with qualitative findings to assess levels of convergence), complementarity (seeking elaboration, illustration, enhancement, and clarification of the findings from one method with results from the other method), initiation (identifying paradoxes and contradictions stemming from a comparison of the quantitative and qualitative findings), development (using the findings from one method to help inform the other method), or expansion (expanding the breadth and range of the study phases).

4.3 Context and Setting of Research

The students for both the Pilot and Main Study were all from an inner-city co-educational K-12 independent school. The participants were chosen from this school as the researcher was also a part-time teacher of mathematics on the high school campus (Years 7-10) and had access to the students.

4.3.1 School Profile

The Index of Community Socio-Educational Advantage (ICSEA) was created by the Australian Curriculum, Assessment and Reporting Authority (ACARA) to provide fair comparisons of National Assessment Program – Literacy and Numeracy (NAPLAN) test achievement by students in schools across Australia. According to their fact sheet (ACARA, 2015), the scale was developed in response to research highlighting key factors in students' family backgrounds (parents' occupation, their school education, and non-school education) that have an influence on students' educational outcomes at school. They also found that school-level factors (geographical location and proportion of Indigenous students) needed to be considered when summarising the educational advantage of a school. ICSEA provides a numeric scale that represents the magnitude of this influence, or level of educational advantage, and considers both student and school level factors (ACARA, 2015). ICSEA values are calculated on a scale which has a median of 1000 and standard deviation of 100. ICSEA values typically range from approximately 500 (representing extremely educationally disadvantaged backgrounds) to 1300 (representing schools with students with very educationally advantaged backgrounds). School profiles, as calculated and determined by ACARA, also show the distribution of students across four Socio-Educational Advantage (SEA) quarters representing a scale of relative disadvantage through to relative advantage. These are calculated using only student-level factors.

There is some debate regarding the veracity of the ICSEA score as "statistically similar" schools may share a similar score but the context, size, demographic composition, and geographical location are often very different. Background data relies on parent self-reporting so it is debatable if this is a valid comparison. It is difficult to make comparisons and deem results transferable to similar students when many factors need to be considered, however ICSEA scores are readily available and provide a baseline comparison provided the limitations are understood.

The school in which the research was conducted had a calculated 2013 ICSEA value of 1167 with a distribution of one percent of students in the bottom quarter, seven percent of students in the lower middle quarter, 18% of students in the upper middle quarter, and 74% of students in the top quarter. There were 1265 full-time equivalent student enrolments across the whole school, comprising 51% boys and 49% girls. One percent of students identified as being Indigenous and 12% of students had a language

Table 4.1

Year 6 Fractions Curriculum

background other than English.

4.3.2 Curriculum

At the time of the study, the current Australian Curriculum was in the process of being implemented across all Australian States and Territories and each education sector within each State had a different timeline for implementation. The school, at the time the research was conducted, had aligned its own learning outcomes to the Australian Curriculum, but each mathematics course was supplemented with additional content to provide students with the best possible knowledge base for their mathematics pathways beyond Year 10. All students from grades K-6 participate in the International Baccalaureate Primary Years Programme (IB-PYP), which focuses on the development of the whole child, providing a framework to meet the academic, cultural, physical, social and spiritual development of each child. The General Capabilities and explicit learning outcomes from the Australian Curriculum are used to support the IB-PYP. Learning content and outcomes specific to fractions for Year 6 IB-PYP are shown in Table 4.1.

Year Level	Units of work involving fractions	Expected outcomes for fractions
6	• Number Learning Continuum • Data Handling Learning Continuum • Programme of Inquiry	• Connect fractions, decimals and percentages as different representations of the same number • Solve problems involving the addition and subtraction of related fractions • Describe rules used in sequences involving whole numbers, fractions, and decimals • Locate fractions and integers on a number line • Calculate a simple fraction of a quantity • List and communicate probabilities using simple fractions, decimals and percentages

Assessment	Based on how learners construct meaning through the assessment of prior knowledge, formative assessment and summative assessment.

The curriculum content for mathematics in Years 7-9 was broadly based and the teaching approach emphasised applications of mathematical knowledge where possible. Problems of many different kinds were presented in a variety of ways to encourage problem-solving skills and a true understanding of the concepts involved. All mathematics courses at the school were developed to ensure the inclusion of the necessary elements outlined in the Australian Curriculum. Allowances were made for the fact students do not arrive at the school with the same mathematical foundation. High school enrolments come from a variety of primary schools not aligned with this particular school. Fractions were taught explicitly in the Year 7 and 8 program, through linear equations in Year 9 and in algebraic fractions in the elective Year 9 Maths Methods course. An overview of the content and expected outcomes can be seen in Table 4.2. Copies of the summative assessments of fraction units for Years 7-9 can be seen in Appendix A.

Table 4.2

Years 7-9 Fractions Curriculum.

Year Level	Units of work involving fractions	Expected outcomes for fractions
7	• Fractions Unit • Fractions, Decimals and Percentages Unit	• Compare fractions using equivalence • Locate and represent fractions and mixed numerals on a number line • Solve problems involving addition and

		subtraction of fractions, including those with unrelated denominators • Multiply and divide fractions and decimals using efficient written strategies and digital technologies • Express one quantity as a fraction of another, with and without the use of digital technologies • Round decimals to a specified number of decimal places • Compare and order integers, fractions and decimals • Connect fractions, decimals and percentages and carry out simple conversions.
Assessment	Year 7 Fractions Test and Year 7 Fractions, Decimals and Percentages Test	
8	• Fractions, Decimals and Percentages	• The four operations with fractions (including improper and mixed numbers) • Finding fractions of an amount • Investigate terminating and recurring decimals • The four operations with decimals (include ×10 and ÷10) • Percentages are /100 • Convert between fractions, decimals and percentages.

Assessment	Year 8 Fractions, Decimals and Percentages Test	
9 (Core)	• Linear Equations	• Solve problems involving first degree equations in which fractions have to be manipulated.
9 (Elective)	• Algebraic Fractions	• Simplify algebraic fractions by cancellation • The four operations of arithmetic applied to algebraic fractions • Solve equations involving fractions (including cross multiplication) • Applications to similar triangles • Application to real world word problems
Assessment	Year 9 Equation Study Test and Year 9 Maths Methods Algebraic Fractions Test	

4.4 Phases of Research

The research was conducted over two years from the time of the pilot study to the time of the delayed retention test. The intervention phase was five weeks in duration, with the two treatment programs running concurrently. The phases of the research are summarised in Table 4.3.

Table 4.3

Overview of the phases of the research.

	Phase	Month	Group
1	Pilot Study	December 2012	Year 9 students (n=128)
2	Fractions Diagnostic Pre-test Self-Efficacy Questionnaire	July-August 2013	Year 7, 8 & 9 students (n=361)
3	Intervention	November - December 2013	O/N (n=20) Traditional (n=15)
4	Fractions Diagnostic Post-test 'a' Self-efficacy Questionnaire	December 2013	Both treatment groups (n=35)
5	Delayed Retention Fractions Diagnostic Post-test 'b'	April – May 2014	Year 8, 9 & 10 students (n=361)

The data were collected through combination of a researcher-constructed instruments. The details of the Fractions Diagnostic Test can be found in section 4.5.5. A copy of the pre- and post-test can be seen in Appendix B, and a copy of Delayed Retention Test can be seen in Appendix C. The reliability and validity of the Fractions Diagnostic Test was verified through a pilot study (details of which can be found in section 4.5). A self-efficacy questionnaire was administered to determine motivation pre- and post-intervention. Details of the self-efficacy instrument, student workbooks, reflection journals, and researcher field notes can be found in section 4.5.5 (copy of the self-efficacy instrument can be seen in Appendix D).

4.5 Pilot Study

4.5.1 Overview

Prior to commencement of the main research, a pilot study was conducted to trial one of the research instruments. In this section the pilot study of the Fractions Diagnostic Test is detailed. A convenience sample of students participated in the pilot study.

4.5.2 Participants

Participants were year nine students in the year the Pilot Study was conducted (n=128). The students were all from the same school as the participants in the main study. The researcher, also a part-time teacher of mathematics at the school, had access to all students from Years 7 through to Year 10, so in planning the project it was decided that the Year 9s (who would be in Year 10 during the period the main research was conducted) would participate in the Pilot Study and the remaining students, newly in Years 7 to 9, would be participants in the main study. Upon consent from the principal, the students and teachers were all given an information sheet about the research and a consent form. All participants indicated their willingness to be involved in the investigation and signed the appropriate consent forms agreeing to have their ability to answer questions regarding understanding of fractions examined. Students undertook the test in an hour-long timetabled mathematics lesson in class groups.

The Year 9 students had all completed explicit teaching of fractions, as detailed in 4.3.2. In Year 7, all operations and problem solving with fractions had been covered. In Year 8, the four operations: mixed number and improper fractions, and conversions between fractions, decimals, and percentages were studied. In Year 9, students electing to take additional mathematics (approx. 60% of the year group) completed a unit on algebraic fractions. The unit on algebraic fractions had been completed at the time of the pilot study.

4.5.3 Overview of Instrument

The pen and paper Fractions Diagnostic Test (FDT) was trialled in the Pilot Study. The final version of the FDT can be seen in Appendix B. Specific items and their purpose for inclusion in the final instrument are discussed in section 4.5.5. The self-developed pilot test instrument comprised 11 question parts, with parts resulting in 48 items in total. Items in the test were inspired by Behr et el.'s (1983) theoretical model linking the different interpretations of fractions to the basic operations of fractions (see section 2.2.2). The questions were directly related to each of the sub-constructs of partitioning, ratio, quotient, measure, equivalence, multiplication, problem solving, and addition. Clarke, Roche, and Mitchell (2008) suggested that mastering each of the interpretations of fractions contributes towards acquiring proficiency. Testing of each of the interpretations can, therefore, help assess an individuals' understanding of fraction concepts.

Based on the work of Byrnes and Wasik (1991) who found that understanding the conceptual aspects of fractions is a prerequisite for procedural ability, the test was designed with explicit conceptual and procedural questions. It was hoped that if repeated procedural errors were made the related conceptual questions might assist in exposing the error pattern. This also supported the theory of Rittle-Johnson et al. (2001) who suggested the "concepts-first" approach where the student is prone to "bugs" if procedures are not understood conceptually. The Fractions Diagnostic Test comprised a mix of conceptual and procedural questions. The design and rationale for the items is given in more detail in section 4.5.5 when the final instrument is discussed.

Conceptual Questions on the Fractions Diagnostic Test included: a comparison of fractions (which is larger, items 1a-i); ordering fractions from smallest to largest (item 2); value of a point on a number line (item 3); marking fractions on a number line showing successive partitioning (item 4; calculating a fraction of a whole number (operator)/fraction of a fraction (items 5a & 5b); worded problems (item 5c & 5d); colouring fraction of a shape (no partitions/partitions – item 5e & 5f); shading a fraction of a circle (5g); equivalent fractions (items 6a-c); simplifying (items 7a-d); draw a shape that shows the whole, given a fractional value (item 9).

Procedural Questions on the Fractions Diagnostic Test included: addition of fractions with the same denominator (item 8a); addition of fractions with different denominators (items 8b & 8d); addition of mixed number fraction (item 8c); subtraction of same denominator (item 8e); subtraction of different denominator (items 8f); subtraction from a whole number (item 8g); multiplication of a simple fraction (item 8h-j); multiplication of mixed numbers (item 8k); fraction division (items 8l-m).

The validity of the test was determined through a pilot study (as described in Section 4.3.1). Validity was maintained through checking the links between fraction knowledge categories and the theoretical model proposed by Behr et al. (1983).

4.5.4 Outcomes

There were 128 students who participated in the pilot study. In what follows, "item number" refers to the number given to each individual item in the FDT. The items that were most problematic for the pilot study students were from Question 8, specifically: 8*i*, 8*j* and 8*k*. These were all multiplication questions.

The errors encountered in question 8*i* where the students were asked to perform the simple fraction multiplication $\frac{3}{5} \times \frac{2}{3}$ were predominantly of two main types: not reducing the answer to simplest form; and finding a common denominator and multiplying the numerator (like the addition algorithm). The results indicated that 50 students did not cancel before multiplying and then did not simplify at the end, although they correctly multiplied numerators and denominators respectively. Twelve of the errors were through finding a common denominator and then multiplying the numerator only. There were also an additional 21 students who did not attempt this question at all.

In question 8*j* the students were asked to perform the simple fraction multiplication, $\frac{6}{7} \times \frac{14}{15}$ that used slightly more complex numbers than 8*i*, but with obvious cancelling. The errors encountered were all a result of not cancelling before multiplying. Of the 64 students with errors (50% of all participants), there were 50 students who did not cancel at all and 5 students who cancelled down the factor of 7 in the 7 and 14 only. Interestingly, only 5 students cancelled before multiplying and got the correct answer. There were 38 students who did not attempt this question at all.

In question 8*k* the question was multiplication of two mixed number fractions, $1\frac{3}{4} \times 2\frac{5}{6}$. Only 7% of the students answered this correctly. The most common error was not converting improper fractions before cancelling, and then multiplying. In an ad hoc interview after the test, only one student indicated that they knew this could also be multiplied using the distributive law. All teachers had taught the rule to convert to an improper fraction first. Ten students (approximately 8%) converted to an improper fraction but did not simplify. A total of 36 students did not attempt this question at all.

Of the 6 students who answered incorrectly the simple fraction addition question involving fractions with the same denominators, all had added both the numerators and denominators together. They answered similar addition questions in the same manner, getting 8*b*, 8*c,* and 8*d* all incorrect.

25 students who answered 8*b*, 8*c* and 8*d* (fraction addition) incorrectly by adding numerator and adding denominator, also answered 6*a*, 6*b,* and 6*c* incorrectly – equivalent fractions.

Of the 70 students who answered 8*i* incorrectly because they did not simplify the fraction $\frac{3}{5} \times \frac{2}{3}$, 27 also answered 7*c* incorrectly, which asked students to simplify $\frac{6}{15} = -$. This perhaps suggests that they remembered the procedure to multiply numerators and multiply denominators but did not know how to simplify the result, did not conceptually understand equivalence, or did not grasp the convention to express fractions in simplest form. Of these same 70 students who answered 8*i* incorrectly, 47 (67%) of those also answered 5*b* ($\frac{1}{3}$ of $\frac{1}{2}$) incorrectly. Table 4.4 highlights the number of errors, percentage of errors, and the number of students who did attempt each item on the pilot test.

Table 4.4

Number and percentage of errors for each item of the FDT

(*n=128*) Item	1a	1b	1c	1d	1e	1f	1g	1h	1i	2	3	4a
No. of errors	23	10	30	17	8	11	8	19	23	46	12	34
% of errors	18	8	23	13	6	9	6	15	18	36	9	27
No. not attempted	0	0	0	0	0	1	2	0	2	0	0	0
Item	**4b**	**4c**	**4d**	**4e**	**5a**	**5b**	**5c**	**5d**	**5e**	**5f**	**5g**	**6a**
No. of errors	22	32	19	53	18	46	48	27	7	47	41	6
% of errors	17	25	15	41	14	36	38	21	5	37	32	5
No. not attempted	0	0	0	0	8	17	18	15	0	2	1	5
Item	**6b**	**6c**	**7a**	**7b**	**7c**	**7d**	**8a**	**8b**	**8c**	**8d**	**8e**	**8f**
No. of errors	18	10	26	4	40	28	7	22	52	47	6	25
% of errors	14	8	20	3	31	22	5	17	41	37	5	20
No. not attempted	5	4	5	5	5	5	3	5	13	8	6	12
Item	**8g**	**8h**	**8i**	**8j**	**8k**	**8l**	**8m**	**8n**	**9**	**10a**	**10b**	**11**
No. of errors	32	36	70	64	83	24	27	44	20	11	18	7
% of errors	25	28	55	50	65	19	21	34	16	9	14	5
No. not attempted	25	25	21	38	36	48	40	39	3	2	17	3

4.5.5 Final Instruments for the Main Study

The Fractions Diagnostic Test trialled in the Pilot Study was modified after feedback and initial analysis for use in the main study. In this section, the rationale behind the design of the items is explained. A copy of the final Fractions Diagnostic Test can be seen in Appendix B.

Questions 1 and 2 – Ordering / Comparing / Benchmarking

Pearn and Stephens (2004) found that some students use whole number gap thinking rather than multiplicative thinking when comparing fractions. Students typically calculate the difference or "gap" between numerator and denominator to compare fractions. Whole number thinking is also behind other strategies where students deal with numerators and denominators individually, ignoring the ratio connecting them. The purpose of comparisons with different denominators is to look for multiplicative thinking, where students adopt strategies to preserve the fundamental ratio between the numerator and denominator. Some of the pairs were deliberately added to the test to ascertain efficiency of comparison by benchmarking, rather than the risk of errors in writing equivalent fractions. Benchmarking, such as to a $\frac{1}{2}$ for example, involves students determining if the numerators are more or less than half the denominators and make the comparison in that way. Other comparison items in Question 1 of the test included comparisons involving equivalent fractions, same denominator, same numerator, denominators that are multiples, and a harder lowest common denominator question. Table 4.5 contains sample items from Question 1 of the test.

Table 4.5

Sample items for Question 1a-1i – comparing fractions.

b	$\frac{5}{8}$ $\frac{7}{8}$	Same denominator
e	$\frac{4}{5}$ $\frac{3}{8}$	Benchmark against $\frac{1}{2}$
f	$\frac{4}{7}$ $\frac{4}{9}$	Same numerator
i	$\frac{5}{7}$ $\frac{3}{4}$	Hard lowest common denominator

Spangler (2011) suggested that when comparing fractions with different denominators, some students make errors in writing equivalent fractions. Other methods of comparison could be benchmarking or by using "cross-products". The cross-product method for fraction comparison involves multiplying the numerator of one fraction by the denominator of the other fraction and then comparing the answers to show whether one is bigger than the other, or if the two are equivalent. Table 4.6 shows Question 2, where students were asked to arrange the fractions from smallest to largest.

Table 4.6

Question 2 – Ordering fractions from smallest to largest

Comparing using equivalent fractions.
$\frac{3}{10}$ $\frac{3}{5}$ $\frac{3}{4}$ $\frac{1}{2}$ $\frac{11}{20}$

Questions 1 and 2 tested the students' knowledge of two components of the Australian Curriculum: Mathematics. The Australian Curriculum suggests students should be able to compare and order fractions with related denominators by the end of year 6 (as per items in Question 2 of the FDT), and by the end of year 7, students should be able to compare fractions using equivalence (Question1).

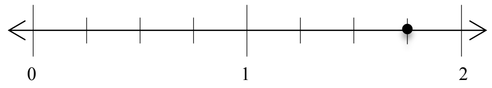
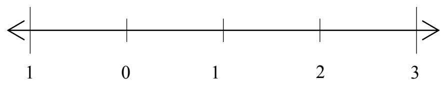
Questions 3 and 4 – Measure

Clarke, Roche, and Mitchell (2007) described how a fraction can represent a measure of a quantity relative to one unit of that quantity. The number of equal parts in a unit can vary depending on how many times you partition. Fractions as units of measure are best demonstrated on a number line, supporting the understanding that a fraction is a number, with a position on the number line. The Australian Curriculum suggests students should be able to locate and represent positive and negative fractions and mixed numbers on a number line by the end of year 7.

A common misconception when using a number line marked with numbers greater than 1 is that some students view the entire distance from 0 to the endpoint as being one unit. Novillis (1976) stated that to test students' understanding of the number line model, a number line longer than one unit should be used. The diagnostic test in this study incorporated two number line questions to assess the *measure* subcomponent of fractions (Table 4.7).

Table 4.7

Questions 3 and 4 – assessing measure interpretation through number line models.

3	 A. $\frac{3}{4}$ B. $\frac{7}{8}$ C. $1\frac{7}{8}$ D. $1\frac{3}{4}$	Value of the point shown on the number line (circle correct value)
4	 $\frac{3}{4}$ $1\frac{1}{3}$ $-\frac{1}{4}$ $2\frac{1}{10}$ $\frac{5}{2}$	Mark these fractions on the number line provided

Questions 5, 9, 10, and 11 – Operator / Quotient / Part-Whole

Question 5 incorporated a range of fraction concept questions to assess knowledge of *fractions as operator*, *quotients* and *part-whole*. According to Behr, Lesh, Post, and Silver (1983) the operator interpretation of rational number is useful in studying equivalence of fractions and the operation of multiplication. Clarke, Roche, and Mitchell (2007) suggested that students' lack of experience in using fractions as operators may contribute to the misconception that multiplication always makes bigger and division makes smaller. A fraction may also represent the operation of division and the consideration of rational numbers as quotients. Items a, b, c, and d in Question 5 explored *fractions as operators and quotient* interpretations (Table 4.8). The Australian Curriculum suggests students should be able to find a simple fraction of a quantity where the result is a whole number by the end of Year 6.

Table 4.8

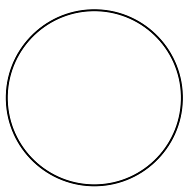
Sample Items for Question 5a-5d – fractions as operators and quotients

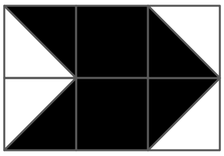
a	$\frac{2}{3}$ of 9	Fraction as operator
c	If there is $\frac{7}{8}$ of a cake left and 14 people would all like a piece, what fraction of a cake will they receive?	Quotient interpretation

Items e, f, and g of Question 5 assessed the students' ability to partition a continuous quantity. According to Kieren (1981), understanding of the *part-whole* sub-construct is fundamental to all later interpretations and is considered to be an important language-generating construct.

Table 4.9

Sample Items for Question 5e-5g, 9, 10, and 11 – Part-whole interpretations

5f	Colour in $\frac{2}{3}$ of this shape: 	Part-whole - difficult
5g	Colour in $\frac{2}{5}$ of this shape: 	Part-whole – students must divide the figure into parts of the same size

9	If this is $\frac{2}{3}$ draw a shape that shows the whole 	Part-whole
11	What fraction of the whole rectangle is shaded? 	Part-whole – some students compare the number of shaded parts to the number of parts not shaded.

Questions 6 and 7 – Equivalent fractions and simplest form

Spangler (2011) suggested that some students use an "additive" method in finding a fraction equivalent to a given fraction. Others may focus on the difference, for whole numbers, between the numerator and the denominator. When writing fractions in simplest form and the fraction has a denominator of 1, some students do not recognise the fraction as the whole number given by the numerator. Questions 6 and 7 of the Diagnostic Test assessed students' ability to find equivalent fractions and to write fractions in simplest form (sample questions in Table 4.10). The Australian Curriculum indicates that students should be able to compare fractions using equivalence by the end of Year 7.

Table 4.10

Sample Items for Questions 6 and 7 – Equivalent Fractions and Simplest Form

6	$\frac{3}{5} = \frac{}{10}, \frac{12}{18} = \frac{}{6}, \frac{1}{3} = \frac{}{18}$	Write these as equivalent fractions
7	$\frac{2}{8} = -, \frac{6}{15} = -, \frac{8}{1} =$	Write the fraction in its simplest form

Question 8 – Procedural computation questions involving the four operations

Carpenter, Fennema, and Franke (1996) argued that an understanding of the way students think about modelling action and relations depicted in problems provides coherence for teaching them how to solve problems involving addition, subtraction, multiplication, and division situations. The Australian Curriculum expects students to be able to perform all four operations of fraction computation by the end of Year 7.

Addition and subtraction of fractions with the same denominator are introduced into the Australian Curriculum in Year 5. In Year 6 students are expected to solve problems involving addition and subtraction of fractions with the related denominators, and by the end of Year 7 be able to solve problems with unrelated denominators. When adding (or subtracting) fractions with like denominators, some students add (subtract) the numerators and add (subtract) the denominators. This misconception of adding the numerators and adding the denominator is also seen for fractions with unlike denominators. It is common for students to remember to find the lowest common denominator but then add (subtract) the numerators without changing the numerators to obtain equivalent fractions. Questions in Table 4.11 are samples of the addition and subtraction components of Question 8 in the Diagnostic Test.

Table 4.11

Question 8 – sample questions of addition and subtraction

8a	$\frac{2}{9}+\frac{5}{9}$	Addition – same denominator
8d	$\frac{2}{3}+\frac{3}{5}$	Addition – unlike denominator
8c	$1\frac{3}{4}+2\frac{1}{6}$	Addition – mixed fractions, unlike denominator
8g	$\frac{6}{7}-\frac{2}{7}$	Subtraction – same denominator
8h	$\frac{9}{16}-\frac{1}{2}$	Subtraction – unlike denominator
8i	$10-\frac{1}{3}$	Subtraction – whole number minus a fraction

The second half of Question 8 in the Fractions Diagnostic Test included items involving multiplication of fractions, multiplication of mixed numbers, and division of fractions (including division involving a fraction and a whole number). The common misconceptions the test was intended to identify were when multiplying two fractions with a common denominator, some students multiply numerators but keep the common denominator. When multiplying mixed numbers, some students multiply the whole numbers and multiply the fractions. Students who know that they are meant to convert to an improper fraction may multiply the whole number portion by the denominator but fail to add the numerator. When dividing fractions, some students rewrite the division as a multiplication but fail to use the reciprocal of the divisor, or they rewrite the division as multiplication but write the reciprocal of the dividend. Other division errors include rewriting the division as a multiplication but multiplying only the numerators or incorrectly simplifying before writing the reciprocal of the divisor. Sample multiplication and division questions from Question 8 of the Fractions Diagnostic Test can be seen in Table 4.12

Table 4.12

Question 8 – sample questions of multiplication and division.

8l	$\frac{3}{5} \times \frac{2}{3}$	Simple multiplication with common factors
8k	$\frac{2}{3} \times 12$	Multiplication of a fraction and a whole number
8o	$1\frac{3}{4} \times 2\frac{5}{6}$	Multiplying mixed numbers
8p	$\frac{3}{5} \div \frac{1}{5}$	Division – same denominator
8r	$\frac{3}{4} \div \frac{1}{8}$	Division – related denominators
8u	$4 \div \frac{2}{3}$	Division of a whole number by a fraction

The Fractions Diagnostic Pre-test and Post-Test ‘a’ were identical instruments. Fractions Diagnostic Post-test ‘b’ (also known as Delayed Retention Test) was designed to emulate post-test ‘a’ in terms of difficulty and the ability to diagnose fraction computation (Appendix C).

Self-Efficacy Questionnaire

In this section, the rationale behind the design of the items in the Self-Efficacy Questionnaire is explained. A copy of the questionnaire can be seen in Appendix D.

Bandura’s Social Cognitive Theory (1986) formed the theoretical framework for the development of the Self-Efficacy Questionnaire used in this study. Bandura stated that perceptions of self-efficacy come from personal accomplishments, vicarious learning experiences, verbal persuasions, and physiological states. Tait-McCutcheon (2008) believed that one way to gain insight into how students feel, think, and act about and toward mathematics is to examine their psychological domains of functioning: the affective, the cognitive, and the conative. The Questionnaire was adapted from the work of Tait-McCutcheon, who examined the self-efficacy of 64 year 4 to 8 students toward their mathematics learning by analysing their responses to affective, cognitive, and conative statements. The adapted Self-Efficacy Questionnaire used in this study included twenty statements divided into the same three psychological domains of functioning.

Consistent with Tait-McCutcheon's questionnaire, the questionnaire in this study consisted of 3 sub-scales, involving 8 affective domain statements, 6 cognitive domain statements, and 6 conative domain statements. Some of the general mathematics statements were changed to reflect the fraction-specific nature of the study and other statements were phrased positively, rather than negatively. For example, an item in the current study was "*I am interested in the things I learn in maths*", rather than Tait-McCutcheon's "*Mathematics does not make sense to me*". A fraction-specific statement in the cognitive domain was "*I like the challenge of a hard fractions problem*". Table 4.13 provides examples of questions from each of the three scales.

Table 4.13

Example statements for each domain from the Self-Efficacy Questionnaire

Domain	Item	Statement
Affective Domain	1.	Working hard leads to success in maths
	7.	I cannot change how good I am at maths
	10.	I know if I am going to get a maths question right
	11.	I enjoy doing fractions
Cognitive Domain	6.	I am interested in the things I learn in maths
	12.	With fractions, I understand even the most difficult work
	16.	I like the challenge of a hard fractions problem
Conative Domain	8.	When I really try I can get through most difficult tasks
	14.	Even when a fraction problem looks hard, I know I can make progress with it.
	19.	If I make a mistake in maths, I try to find out where I went wrong.

Students indicated their response for each item on a 5-point Likert scale ranging from 1 (strongly disagree) to 5 (strongly agree). Six items (3, 5, 7, 9, 13, and 15) were reverse scored before analysis as they were negatively worded.

4.6 Main Study

On completion of the Pilot Study, and after the refinement of the instruments, the main study was conducted. A total of 361 students completed the pre-test, which was all students from the study school who were studying in the high school years, and who did not participate in the Pilot Study. The pre-test instruments included both the Fractions Diagnostic Pre-Test and the Self-Efficacy Questionnaire. After analysis of the diagnostic pre-test, 83 individual students were identified were invited to participate in the intervention programs. Participants were randomly assigned to one of the two intervention groups. After intervention, participants then completed the Fractions Diagnostic Post-Test "a" and a repeat of the Self-Efficacy Questionnaire. All participants who completed the pre-test Fractions Diagnostic Test were then invited to participate in a Delayed Retention Test, Post-Test "b". Analysis of results for all instruments was conducted throughout the testing phases.

4.6.1 Pre-tests

All students from Years 7-9 in the study school were invited to participate in the main study. Students from Year 10 were excluded from the research as they had participated in the Pilot Study the previous year (see details in section 4.5). Following consent from the principal, the students and teachers were all given an information sheet about the research and a consent form (Appendix E). All participants indicated their willingness to be involved in the following facets of the investigation and signed the appropriate consent forms. They agreed to:

a. have their ability to answer questions regarding understanding of fractions examined;

b. have their attitudes and beliefs about their ability to do fractions examined;

c. participate in an intervention programme, if invited;

d. share work samples and reflective journals; and

e. participate in ad-hoc interviews

Sixteen class groups of students participated in the initial part of the study. The groups consisted of: 6 classes of Year 7 (average age of 13 years at the commencement of the study), 5 classes of Year 8 (average age of 14 years) and 5 Year 9 classes (average age of 15 years). The classes averaged 22 students. The students in Year 7 were not ability streamed; therefore, each group represented a range of ability levels. In Year 8 and Year 9 there was one class from each year level that was streamed into a lower ability group and who were taught a modified mathematics curriculum. Table 4.14 shows the participant numbers by year group and gender for the diagnostic pre-test and self-efficacy questionnaire administered at the beginning of the study.

Table 4.14

Number of students by year group and gender for the pre-test and self-efficacy questionnaire.

Year Group	Male	Female	Total
Year 7	67	66	133
Year 8	55	68	123
Year 9	55	50	105
Total	177	184	361

Fractions Diagnostic Test.

The purpose of the pre-test was to expose repeated errors in computation and was used for diagnostic purposes. The pre-test was used to get an indication of the nature and extent of students' difficulties with fractions, and to identify a cohort with particular difficulties who might be amenable to an intervention. Students who applied the wrong method of computation two or more times to similar questions were identified as having a repeated error and were invited to participate in the intervention program offered by the study. For example, if a student added both the numerators and the denominators in addition questions, with same and/or different denominators, this was considered a repeated error. Similarly, students who routinely multiplied whole numbers separately in mixed fractions without converting to an improper fraction were also identified as having a repeat error in computation. Overall individual results were also recorded for quantitative analysis of an inferential nature in comparison to the post-test and delayed retention test. Results were recorded in terms of both whether the response to each question was correct or incorrect and the method that was used to answer the question. Participants were asked to include all working out and some questions had an area to record reasons for answering the question in the way they did. Understanding and improvement of fraction computation was determined through the method the student used to answer the question. An individual's overall score for each test was recorded for comparison between the tests; however, this was not a focal point of analysis.

Self-Efficacy Questionnaire

All participants completed the self-efficacy questionnaire (Appendix D) prior to any intervention to assess their confidence and attitude toward mathematics in general and in working specifically with fractions.

The students responded to each statement by selecting either: strongly disagree, disagree, neutral, agree, or strongly agree. The questionnaire was completed before the Fractions Diagnostic Test so that any test anxiety did not affect the students' self-efficacy. Students indicated their degree of agreement to statements such as, "I often get mathematical questions wrong but I do not understand why".

Selecting participants for the intervention program

After the diagnostic pre-test was analysed, 83 individual students were identified as having repeated errors in fraction computation, and were invited to participate in the intervention programs.

4.6.2 Intervention Programs

Forty students accepted the invitation to participate in the intervention program, with a total of 35 completing all aspects of intervention and post-testing (15 in the Traditional Group and 20 in the O/N Group). The students completed five weeks of intervention and then completed the Post-Test 'a' and the repeat Self-Efficacy Questionnaire. Students were randomly assigned to one of the two intervention groups. Table 4.15 shows the participant numbers for the Traditional Intervention Program by year group and gender. Table 4.16 shows the participant number for the O/N Intervention Program by year group and gender.

Table 4.15

Number of students in Traditional Intervention Program by year group and gender

Year Group	Male	Female	Total
Year 7	5	4	9
Year 8	3	2	5
Year 9	1	0	1
Total	9	6	15

Table 4.16

Number of students in O/N Intervention Program by year group and gender

Year Group	Male	Female	Total
Year 7	6	7	13
Year 8	1	3	4
Year 9	1	2	3
Total	8	12	20

Selection of Students for the Intervention Programs

For the purpose of the study as a means of selecting participants for intervention, a repeated error was characterised by two or more questions of the same type being incorrect. Questions requiring procedural computation, including two fraction addition questions and two fraction multiplication questions, were used as the screening questions. After analysis of the pre-test, participants who were found to have repeated errors in fraction computation in two or more questions (n=83) were invited to participate in an intervention program to assist them with their misconceptions. A spreadsheet of students' responses, including misconceptions, was created and each individual's errors were coded for reference in designing the remediation work. It was these codes that the treatment programs were based on. For example, if a student added both the numerator and denominator without finding a common denominator in fraction addition, a code of AN/AD was given.

The two intervention programs offered traditional remediation and the O/N treatment method, respectively. Participants (n=40) were randomly assigned to one of the two treatment groups. The students who declined the invitation to participate in an intervention program (n=43) were considered a control group as they were identified as having repeat errors but did not receive any treatment. Their results for the pre-test and delayed retention test were used for comparison in the analysis of the effectiveness of the treatment programs.

As the participants came from a range of year levels, from Year 7 to Year 9, in order to have access to all participants as a group, the treatment programs were conducted during the students' lunch break of 40 minutes duration. The programs were conducted in adjoining classrooms in the school's mathematics block. The intervention programs ran for two sessions each week for five weeks, with a total of ten sessions for each group. The researcher conducted the O/N sessions as the researcher had knowledge of and experience with the method. Another classroom teacher conducted the traditional remediation sessions. As the classrooms were adjacent, the teacher was able to clarify any questions regarding instruction and the researcher had the opportunity to observe some of the teaching and learning during each session. It is acknowledged that there were limitations to teacher approach and effectiveness with the intervention sessions being conducted simultaneously requiring there to be an additional teacher involved in delivering the traditional remediation program. Due to time constraints and having limited access to the students, it was essential to conduct the sessions at the same time. The researcher and teacher met after each session, and lesson content and plans were shared on internet-based collaborative documents.

As the intervention participants were a mix of students from Years 7-9 and the intervention sessions were conducted during lunchtime, there was minimal conflict regarding the researcher also being a part-time teacher at the school. The teacher taught only one of the O/N intervention participants and remained inquiry oriented throughout the intervention.

The Old Way / New Way Intervention

The O/N methodology prescribes a sequence of four steps that are followed exactly, in full, and in the correct sequence. The four steps of the O/N are reactivation, labelling, discrimination, and generalisation. In step 1, *reactivation of the error memory*, the student was asked to complete the fraction addition or multiplication problem in their usual way, for example $\frac{2}{3} + \frac{3}{5}$. For step 2, *labelling and offering an alternative*, the student was asked if that particular method of performing the computation could be called the "old way". When the student consented, the student was asked if a "new way" for computing $\frac{2}{3} + \frac{3}{5}$ could be demonstrated. The differences between the two computations were then pointed out usually carefully selected language. In step 3, *discrimination*, the student was asked to perform the computation the old way, then the new way, and then asked to contrast the two ways. This discrimination of the same problem ($\frac{2}{3}+\frac{3}{5}$) was repeated five times. In step 4, *generalisation*, the student was given six fraction addition questions and asked to complete them using the new way. This set of steps comprised one complete O/N trial, which took approximately 10 minutes.

There were between 15-20 students attending the O/N treatment program at each session so the researcher set up each student's workbook prior to each lesson. The students were asked, collectively, to complete the question at the top of their page using their own method. As the students completed the question the researcher went around checking their responses and, if incorrect, engaging subsequent steps in the process for each individual. Figure 4.1 shows examples of work from student workbooks.

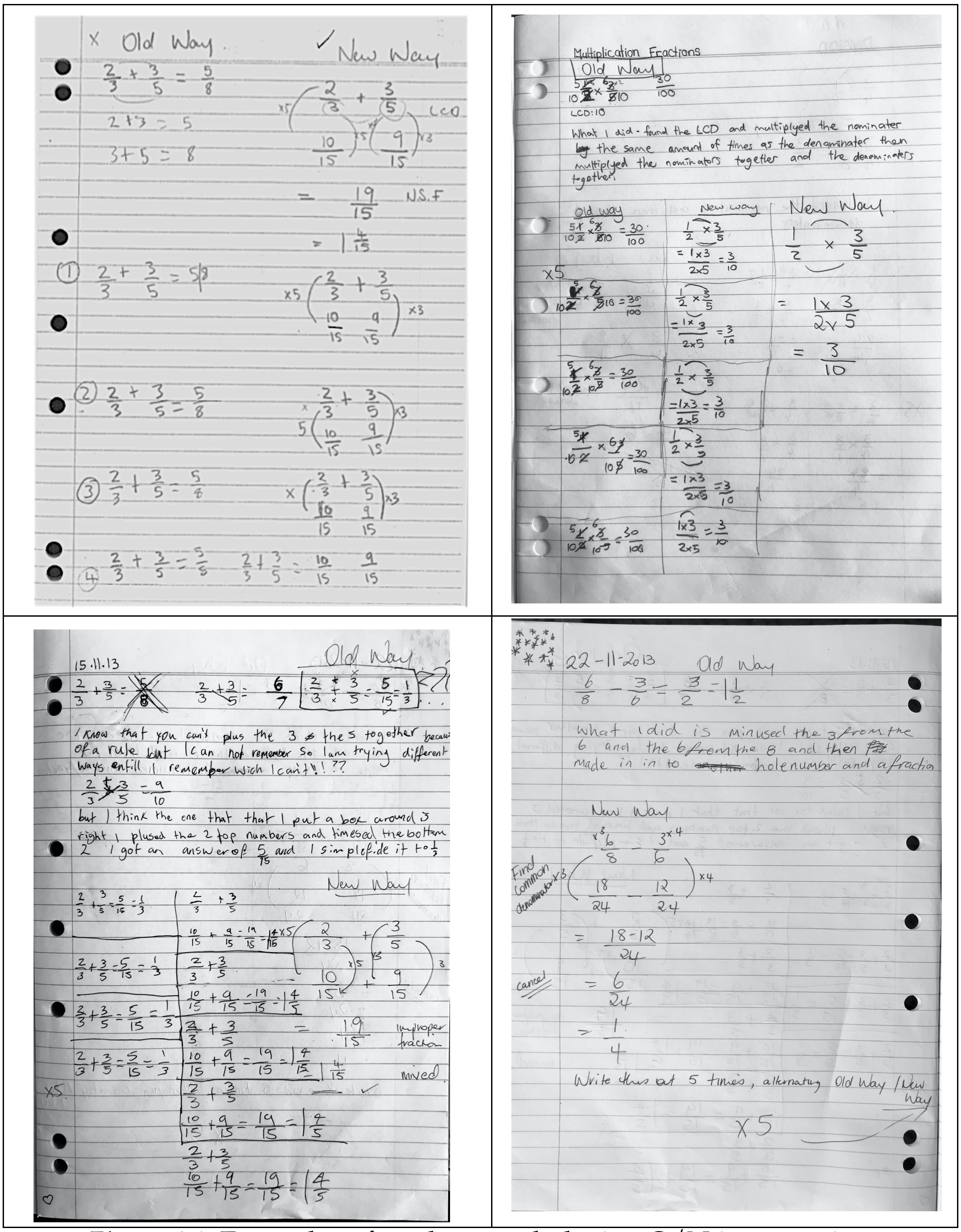


Figure 4.1. Examples of student work during O/N intervention

Further work examples can be found in Appendix F.

For the *generalisation* stage, the researcher had prepared individual questions in each workbook for the student to work on. As there were a range of ages and ability levels within the group, the students all worked at different paces, making it easier for the researcher to get around each student. As a student finished one trial, they wrote about what they had learnt in the lesson in their reflective journal, handed their workbook back in, and left. The researcher checked the workbooks after each session and tailored questions to the individual for the next session. Each trial and journal entry took between 10 and 20 minutes, depending on the individual.

Traditional Intervention

Students in the traditional intervention program participated in a re-teaching program that focused on the conceptual aspects of fraction learning. The students participated in a 25-minute session twice per week for 5 weeks. The lessons were based on the pre-test results, where the teacher and researcher had the opportunity to discuss individual results and misconceptions and design a program accordingly. The lessons were a mixture of instruction and practice. Instruction was based on building knowledge through concrete representation, symbolic representation, abstract application, and problem-solving strategies. The whole class instruction initially focused on the four key concepts of common fractions: the whole or '1', the denominator, the numerator, and a fraction being a number and having a position on a number line. The order of topics covered was:

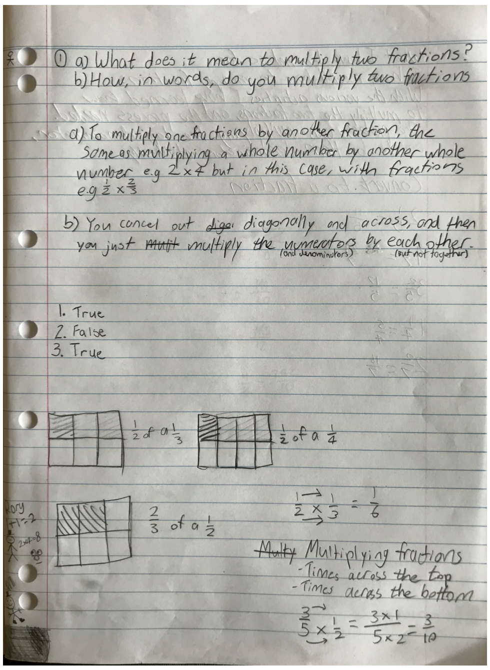
1. Equivalent Fractions – different names for the same quantity
2. Same denominators – can only add the same "type of thing" (same base pieces)
3. Different denominators – need to have same "type of thing"
    - Least common denominator

- Renaming by equivalent fractions
- Simplifying
- Improper fractions/mixed numbers

4. Alternative method for adding fractions $\frac{a}{b} + \frac{c}{d} = \frac{(ad + bc)}{bd}$

Students had a workbook and reflective journal that they completed all work in, and stored worksheets and a reflective journal in which they wrote a reflection after each session. As the traditional remediation sessions lasted longer than the O/N sessions, the researcher was able to observe some instruction time and conduct informal interviews with the students about their work.

Figure 4.2 shows examples of student work from the Traditional Intervention Group. Lesson plan outlines and further work examples can be found in Appendix F.



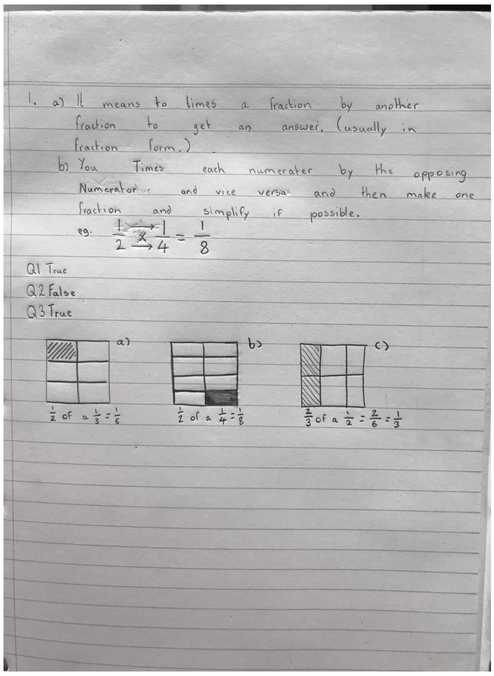


Figure 4.2. Examples of student work during Traditional intervention.

4.6.3 Post-test 'a'

Post-test 'a' was a duplication of the pre-test: a 55-item paper, consisting of 27 conceptual items and 28 procedural items (see Appendix B).

4.6.4 Delayed Retention Test

Post-test 'b' (or the Delayed Retention Test – see Appendix C) was constructed to emulate post-test 'a'. The layout and question type were replicated with only the specific values of fractions in each question changed. For example: question $8a$ $\frac{2}{9} + \frac{5}{9}$ on post-test 'a' was aligned with question $8a$ $\frac{2}{7} + \frac{3}{7}$ on post-test 'b'. Both questions were checking for understanding of addition with the same denominator. They were both simple fractions and deemed to be equal in difficulty level.

For the Delayed Retention Test, the participants were from the same cohort of students who participated in the diagnostic pre-test the previous academic year. The students had all moved up to the next academic year group and were in Years 8, 9 and 10. Students were now aged between 13 and 16 years. From the 361 students who participated in the pre-test, 34 of those students were absent for delayed post-testing.

At the time of the Delayed Retention Test, all participants had been taught fractions in the normal school curriculum. Year 7 students' instruction included concept of fractions; simple fractions, equivalent fractions; comparison of fractions, addition and subtraction of fractions with common denominators; proper fractions; improper fractions and mixed numbers; addition and subtraction of fractions with different denominators; multiplication and division of fractions; fractions of a quantity; expressing one quantity as a fraction of another; simple word problems on fractions and decimal fractions. The Year 8 students included conversion between fractions, decimals and percentages and the Year 9 students had also learnt algebraic fractions at the time of testing. It was beyond the scope of this research to determine the effect of ongoing instruction prior to the delayed response as it would have been unethical to withhold normal instruction from any of the participants. This interaction was considered and will be discussed in section 6.3.

4.7 Other Data

4.7.1 Researcher Data

Researcher developed field notes

During the intervention programs a daily journal of events for each lesson was compiled using a shared electronic document. As the two intervention programs ran concurrently, the researcher and the other teacher were able to document and share notes using a live document. Notes of each teaching session were transcribed in detail at the earliest convenience to provide an overview of each lesson.

Ad hoc interviews

At the conclusion of every teaching session, the researcher discussed the lesson with the classroom teacher assisting with the program. Through this informal interview situation, the researcher gathered data from an observer's perspective. The researcher also conducted informal interviews on an ad hoc basis with the students at various times. This enabled the researcher to check the progress of each student and to record feedback from the student about how they approached each question.

4.7.2 Participant-generated data

Student workbooks and journals

Wilcox and Munroe (2011) suggested that mathematical understanding is enhanced by encouraging students to reflect on their mathematics lessons through writing. Wilcox and Munroe's suggestions were incorporated into this research as a data gathering technique, but also for increasing the students' self-reflective skills. All participants in the intervention programs were given a workbook, in which they completed their work in the front of the book and wrote self-reflections at the back of the book. The students were given some guiding sentences to complete at the back of their book after each lesson: *Today I learned…. I think I am getting better at… I am still confused about….*

4.8 Methods of Data Analysis

Two major forms of data analysis were pursued in this mixed methods research. The design can be described with a "QUAN/qual" notation, indicating that the study was primarily quantitative but also used some qualitative data to supplement the quantitative findings (Creswell, 2009). The quantitative data from the administration of the questionnaires and the diagnostic tests were collected and scored. Where simple descriptive statistics were required, Microsoft Excel Visual Basics programming was utilised. Where statistical significance needed to be ascertained, the statistical package SPSS was used to conduct paired sample t-tests.

The qualitative data were collected through informal interviews, field notes, work samples, artefacts and reflective journals. These were all in written form. The researcher used the data to engage in concurrent data triangulation (Creswell, 2009; Yin, 2009) where the quantitative and qualitative data were compared to identify convergence, inconsistency, or contradiction. This was determined to be useful in validating the data to explain any variance. Bouchard (1976) stated, the convergence "… enhances our belief that the results are valid and not a methodological artefact" (p. 268). Literature from Creswell (2009) suggests that no measures are perfect and that multiple measures in a study share theoretically relevant components. This reduces the uncertainty of the interpretation.

All of these data were then carefully scrutinised so as to identify and make sense of themes that emerged. The initial analysis of the data generated a number of findings that were interpreted, drawing on the literature to generate assertions. The assertions formed the basis of the conclusions that answered the research questions.

4.9 Ethical Considerations

An application to undertake this research was first approved by HREC (Tasmania) Network in 2012. Informed written consent was obtained from: the principal of the school; the parents/guardians of the participants; the participants; and the mathematics teachers (Appendix E).

All participants in the research were informed of the exact nature of their involvement, the expectations required of them, and the resultant time commitments. All participants were given assurances regarding the confidentiality of the research data and the anonymity of the reporting of data. Participants were assigned a reference number; however, it was also essential for participants to include their name on the pre-test for re-identification purposes. Participants were referred to by a reference number only after the intervention phase. Participants were also informed that they were free to withdraw at any stage and that if at any point they wished the data already collected to be removed from the study, this would be done. The collected data will be securely stored for at least five years post the publication of any papers.

4.10 Summary

In this chapter, the design of the research was described. The design of the research was to use mixed methods to conduct an intervention study using an experimental group design with extensive quantitative pre- and post-intervention data, supported by some qualitative aspects. The focus of this chapter was to describe the trial of the newly-developed Fractions Diagnostic Test and the mathematics Self-Efficacy Questionnaire, focussing on student beliefs about their ability to do fractions. After the Pilot Test, the Fractions Diagnostic Test was used to identify the difficulties students have when working with fractions. The difficulties in fraction understanding were discussed in terms of the construct of fractions and participants with misconceptions (repeat errors) were chosen for participation in the intervention programs.

In the next chapter, the common misconceptions held by students about fraction computation was examined.

Chapter 5

Results and Discussion: Fraction Understanding

In Chapter 2 the difficulties students have in understanding fractions were discussed and common fraction misconceptions identified from past studies were highlighted. Difficulties in fraction understanding were discussed in terms of the construct of fractions. Chapter 3 proposed the notion of the interference effect of prior learning on fraction understanding and gave a possible explanation for the common misconceptions occurring in the work of high school mathematics students. In this chapter, the data gathered using the methodology discussed in Chapter 4 are analysed and interpreted. The theoretical frameworks introduced in Chapter 2 are used to define problems encountered by Years 7, 8, and 9 students in learning fractions. The data are examined in terms of past research to confirm if the results of this study support past research findings and are used to identify any additional issues students in this research have with fraction computation. The data source for the findings in this chapter was the fractions diagnostic pre-test, administered to 361 students in years 7, 8, and 9.

The aim of this chapter is to examine deeply the common misconceptions held by students about fraction computation by considering the difficulties encountered by high school students when administered a fractions diagnostic test after the core knowledge for fractions, prescribed in the Australian Curriculum, has been delivered. The test consisted of questions to examine both conceptual and procedural understanding and was comprised of questions linked to the sub-constructs of Behr et el.'s (1983) theoretical model.

Conceptual understanding was examined through questions related to the part-whole, operator, quotient, and measure sub-constructs. For analysis, fraction comparison tasks and questions related to measure were grouped together as "ordering". For the fraction comparison tasks, students had to use the relationship between the numerator and denominator to compare the relative size of two fractions, place a fraction on a number line, and sort a series of fractions in ascending order.

Procedural understanding was determined with respect to the four operations of fractions: addition, subtraction, multiplication, and division. Procedural competence was also examined through questions of equivalence and simplification. The 55-item test (Appendix B) consisted of 27 conceptual questions and 28 procedural questions.

The first section of the chapter will report the overall performance of all participants. A further four sections (5.2-5.5) examine individual items in more detail, based on the four categories of ordering, part-whole/partition, equivalence/simplification, and, finally, procedural operations.

Mean percentage scores for each year level were used to highlight the understanding of each concept or procedure. This chapter concludes with a summary of the most common misconceptions of fraction understanding, highlighting those that are well known by researchers and those unique to this study.

5.1 Overall Performance on the Fractions Diagnostic Test

The 361 students who participated in the pre-test came from sixteen different classes across the high school. Each group of students was in a class of approximately 22 students. The groups consisted of: six classes of Year 7 students (average age of 13 years at the commencement of the study), five classes of Year 8 students (average age of 14 years), and five classes of Year 9 students (average age of 15 years). The students in Year 7 were not ability streamed; therefore, each group represented a range of ability levels. In Year 8 and Year 9 there was one class from each year level that was streamed into a lower ability group and taught a modified mathematics curriculum.

Three out of 361 participants scored 100% on the 55-item pre-test. A further five participants used appropriate methods, and obtained correct answers but did not write one or more fraction answers in the simplest form. For the purpose of analysis, if the student did not write the fraction in the lowest terms it was classified as the error "not simplest form" (NSF). Including those students who did not simplify their answers, this means only two percent of students successfully answered all questions. Scores were generated by 1 mark per question on a right/wrong basis. As the research was primarily examining repeat error methods for fraction computation, for an answer to be given a score of 1 it had to be devoid of errors. No credit was given for partially correct answers, instead a code was assigned to the response, as described in section 4.6.2. Overall mean scores (out of 55) and standard deviations were calculated for each year level and are presented in Table 5.1. Although the mean scores indicate averages of between 65-75% success across the year groups, the standard deviation for each level highlights the range of results, representing scores of zero through to the possible 55.

Table 5.1

Descriptive statistics for the pre-test scores for each year level showing mean (out of 55), standard deviation and mean percentage correct.

Group	Mean	Std. Deviation	Mean %	N
Year 7	36.51	11.79	66	133
Year 8	41.67	9.76	76	123
Year 9	39.10	12.2	71	105
Whole cohort	39.03	11.4	71	361

These results are higher than those achieved on a test completed in a representative study by Siegler and Pyke (2013), which found eighth grade students achieved an average accuracy of 57% on fraction arithmetic problems that included all four operations and operands with numerators and denominators of five or less. Students from Siegler and Pyke's study were recruited from three low-income public school districts in Pennsylvania, U.S. This was in contrast to the students in this study, who were from an independent school in Australia with students likely to have a higher socio-economic background. The difference in mean scores across the year levels reflects the students' fraction understanding based on curriculum content covered. The higher mean score in Year 8 (M = 41.67), compared to that of Year 7 (M = 36.51) can be attributed to the explicit teaching of all fraction operations and fractions, decimals and percentages in Year 8. Students' results in Year 9 (M = 39.10*)* were slightly lower than that of the Year 8s, probably as a result of them not receiving explicit teaching of fractions since Year 8 and the usual decline in performance associated with time passing after explicit teaching.

There was a marked difference between students' mean values for those in the "modified" mathematics classes compared to those in the core classes., as seen in Table 5.2. The students in both the Year 8 modified class and the Year 9 modified class had significantly lower mean scores compared to their peers studying the core curriculum. There was a significant difference between the mean score of the Year 8 modified class compared to the mean score of the Year 8 core classes (p = .000). There was also a significant difference between the mean score of the Year 9 modified class compared to the mean score of the Year 9 core classes (p = .001).

Table 5.2

Descriptive statistics for the pre-test scores for the modified mathematics class in comparison to core class results.

Group	Mean	Std. Deviation	Mean %	N
Year 8 Modified	19.43	6.97	35	7
Year 8 Core	43.02	8.10	78	116
Year 9 Modified	26.56	12.64	48	9
Year 9 Core	40.28	11.44	73	96

Siegler and Pyke (2013) found that the performance difference between students displaying low fraction knowledge compared to those achieving high fraction knowledge was much greater in Year 8 than Year 6. Their study involved sixty Year 6 students, average age of 12, and sixty Year 8 students, average age of 14. The study demonstrated a widening of the gap between the performance of the lower achieving students and their peers as they got older. The results from this study indicate there was a gap in fraction knowledge between the students in the modified curriculum class compared with peers in the core mathematics classes. The difference in the Year 8 modified students' results in comparison to their peers was much greater than for the Year 9 modified students. One explanation for this may be that the Year 8 modified class follow the same sequence as their peers in the core classes, they just do not have the same depth of inquiry into each unit. In Year 8 this means that students are not necessarily receiving targeted curriculum that addresses gaps. The Year 9 modified class has a different sequence of mathematics curriculum to their peers.

When examining the procedural and conceptual understanding of the fraction construct, all students performed better for questions about ordering and simplification/equivalence than the other categories (as can be seen in Table 5.3). Students in Year 7 got the lower scores for arithmetic operations. Students in Year 7 had all completed a dedicated fractions unit within their mathematics classes. For some students, this was the first time they had encountered the algorithm for division of fractions. Surprisingly, the Year 8 students performed better than the Year 9 students in arithmetic operations. This may be attributed to the explicit teaching of fraction operations in Years 7 and 8, as part of the fraction, decimals, percentages units in both year levels. Students in Year 9 core classes had not had any explicit teaching of fractions during the year of testing. Students in the Year 9 modified class (n=9) had done some revision of fraction operations at the beginning of the year. A subset of Year 9 students had algebraic fractions in their curriculum, as they had chosen a mathematics elective class. The algebraic fractions unit in the elective course was scheduled to take place after the research testing, so it would not have influenced students' understanding of fraction operations at the time of testing.

Table 5.3

Mean percentage of correct responses and standard deviation for each sub-construct.

	Conceptual Questions		**Procedural Questions**	
	Ordering	Part-Whole	Equivalence/ Simplification	Operations
Year 7	82 ± 24	68 ± 25	81 ± 30	49 ± 62
Year 8	89 ± 16	78 ± 25	88 ± 23	61 ± 44
Year 9	85 ± 21	72 ± 34	81 ± 34	57 ± 53
Whole cohort	85 ± 22	72 ± 33	83 ± 30	55 ± 54

A correlation analysis was used to assess the relationship between conceptual (ordering and part-whole) and procedural (equivalence/simplification and operations) categories. The correlation analysis revealed that conceptual categories correlated significantly with each other (Table 5.4). They also correlated positively with procedural categories.

Table 5.4

Correlation analysis between conceptual and procedural categories.

	Conceptual Questions		**Procedural Questions**	
	Ordering	Part-Whole	Equivalence/ Simplification	Operations
Part-Whole	0.65			
Equiv/Simp	0.92	0.89		
Operations	0.61	0.68	0.70	

An ANOVA for repeated measures was conducted with category as within-subjects factor (ordering; part-whole; equivalence/simplification; operations) and year group as a between-subjects factor. There was a significant grade effect, $p < 0.001$. A post-hoc test revealed a significant difference in fraction understanding between Years 7 and Year 8.

According to Hiebert and LeFevre (1986) there is a relationship between students' understanding of part-whole relationships and their achievement with fractions. As can be seen in the results, there was a strong positive correlation between part-whole items, ordering items, and the procedural items of equivalence/simplification and operations.

The next sections will examine the students understanding of the sub-constructs more closely.

5.2 Ordering

The first four questions of the pre-test examined students' comprehension of the structure of rational numbers. Students had to use the relationship between the numerator and denominator to compare the relative size of two fractions, place a fraction on a number line, and sort a set of fractions into ascending order.

5.2.1 Comparison of fractions.

On question 1 of the test, students had to choose which of two fractions was larger, for a set of comparisons. Each comparison was designed purposefully to determine/diagnose the students' ability and potential techniques for comparing fractions. There were questions which could be benchmarked against 1/2; questions with the same denominator; questions with the same numerator; questions involving equivalent fractions; a question where the denominator was a multiple of the other; a question where both fractions were one "piece" away from one, and questions that had to be determined by using the lowest common denominator.

Table 5.5

Percentage of correct responses for Question 1, where students were asked to circle the larger fraction, or write = between them.

	a. $\frac{5}{8}$ $\frac{1}{3}$	b. $\frac{5}{8}$ $\frac{7}{8}$	c. $\frac{2}{3}$ $\frac{3}{4}$	d. $\frac{4}{6}$ $\frac{2}{3}$	e. $\frac{4}{5}$ $\frac{3}{8}$	f. $\frac{4}{7}$ $\frac{4}{9}$	g. $\frac{5}{7}$ $\frac{3}{8}$	h. $\frac{3}{10}$ $\frac{2}{5}$	i. $\frac{5}{7}$ $\frac{3}{4}$
Year 7	82	95	78	76	94	89	92	90	77
Year 8	93	97	85	89	98	93	93	91	83
Year 9	88	97	75	86	95	90	95	93	84
Whole cohort	87	96	80	83	96	91	93	91	81

Although the mean percentages were relatively high for all items, the low performance on comparison of the common fractions, in item *c* of Question 1, was concerning. This item returned the lowest percentage of correct responses, and even more concerning was that the Year 9 students had the lowest percentage of correct responses with only 75% of students answering this correctly. One possible explanation for students getting this incorrect is if they have used "gap thinking". This method involves calculating the difference (gap) between the denominator and the numerator of each fraction, and then comparing the two differences to identify the difference that is larger. Spangler (2011) suggested that an error may occur if students use the gap method and determine there is the same gap, of one, between the numerator and denominator and conclude the fractions are therefore equal. Two incorrect conclusions are possible with gap thinking on this question: as each difference between the numerator and denominator is one, the fractions must be equal; or, alternatively, erroneously reasoning that "the smaller the denominator, the bigger the pieces" and so the 2/3 is deemed bigger than 3/4. Of the 72 students who answered this question incorrectly, 38% of incorrect responses stated that these fractions were of equal value. The other 62% of incorrect responses may be attributed to students using the gap method and then comparing the relative size of each gap piece. Students may incorrectly conclude that 2/3 is bigger than 3/4, as pieces from a whole divided into three parts are bigger than pieces from a whole divided into four parts. Students who use whole number gap thinking would, in that case, need to consider that because fourths are smaller than thirds (for unit wholes of the same size) and both fractions are each one part away from being a whole unit, ¾ is closer to being a whole than is 2/3.

Of the 61 students who answered Question 1 (d) incorrectly, 68% believed 2/3 was bigger than 4/6. Students who apply the strategy "the bigger the denominator the smaller the fraction" are demonstrating incomplete strategies. In these cases, students disregard the numerator and, hence, the number of pieces involved. Students may also have used "gap" thinking to conclude that $\frac{2}{3}$ is closer to 1 as it is only one piece away compared to two for $\frac{4}{6}$. Vinner (1997) described this as pseudo-analytical behaviour. This demonstrates some, but not complete, understanding.

One of the most difficult comparisons was between $\frac{5}{7}$ and $\frac{3}{4}$, item *i*, as the two fractions are very similar in size. For most students, successful evaluation of this comparison would require the construction of appropriate equivalent fractions, involving a common denominator. Of the 104 students who got this question wrong, only two of these tried to convert to equivalent fractions for comparison but then made a multiplication error. Fifty percent of incorrect responses indicated that these two fractions were of equal value. If these students tried to estimate their equivalence, or did a mental calculation without converting these to fractions with common denominators, they could incorrectly conclude they were of equal value due to them being very similar in size. It was difficult to determine the method used for the remaining incorrect responses. Students may have used either a whole number scheme and concluded both 5 and 7 are bigger than 3 and 4, therefore making $\frac{5}{7}$ a bigger fraction, or it was simply a guessed response (Pearn & Stephens 2007).

In a similar study conducted by Gabriel et al. (2013) students were asked to compare the size of fractions by indicating which of two fractions was larger. Participants in the study were 292 Belgian students, aged between 10 and 11 years. They found that comparisons using the same denominator yielded better mean percentage scores (Mean = 83 $\pm$ 2%) than those involving the same numerators (Mean = 56 $\pm$ 2%) and those that did not have either the same denominator or numerator (Mean = 65 $\pm$2%). This study supports the findings of Gabriel et al.'s study with 96% accuracy on item *b* with the same denominator, yet 81% accuracy on item *i* which has no common components. Students in this study performed better on item f, where the fractions had the same numerator, with 91% success.

Shortcomings in students' understanding of fractions as numbers with magnitude were confirmed by the poor percentage of correct responses for Question 1. An outcome of the Australian Curriculum is the expectation that students understand fractions as numbers with magnitudes that can be compared and ordered. As Ni and Zhou (2005) also concluded, students appear to be influenced by whole, or natural, number bias when comparing fractions. As discussed in section 2.3.1, one of the main difficulties is that students encounter fractions after they have established ideas and procedures for natural numbers.

5.2.2 Put fractions in ascending order.

Table 5.6 presents the percentage mean response for Question 2 where students were asked to sort a list of fractions into ascending order.

Table 5.6

Mean percentage of correct responses to Question 2.

Arrange in order from smallest to largest $\frac{3}{10}, \frac{3}{5}, \frac{3}{4}, \frac{1}{2}, \frac{11}{20}$	
Grade level	Percentage correct
Year 7	56
Year 8	68
Year 9	57
Whole cohort	60

Common errors were to put the items in reverse order and ordering solely by size of denominator. Reverse order demonstrates a possible understanding of the size of the fractions in relation to each other, with either a misconception of magnitude or by simply misunderstanding whether the question was asking for ascending or descending order. Ordering by size of the denominator demonstrates a whole number bias in fraction magnitude. These students have made a judgement based only on considering the denominator as a whole number, and are unable to think of fractions as integrated magnitudes (Schneider & Siegler, 2010). Students who ordered by size of denominator (Figure 5.1) with ½ in the middle, demonstrated a comparison of the whole number and a bias of a half having to be the midpoint.

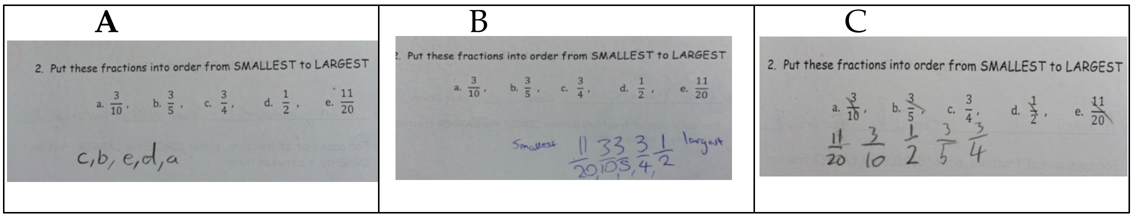


Figure 5.1. Illustrations of common errors when asked to place five fractions in ascending order. Reverse order (**A**) and by size of denominator (B). Size of denominator, with ½ in the middle (**C**).

The findings in this study supported the findings of Brown and Quinn (2006) who found only 57% of senior high school students could order a set of three fractions into ascending order. More than half of the incorrect responses in their study were in reverse order.

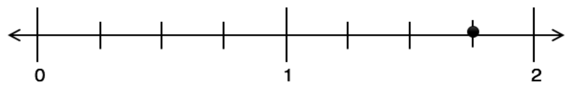
5.2.3 Identify the value of a fraction positioned on a number line.

The question asking students to identify a fraction positioned on a number line (see Table 5.7) was generally done well. Table 5.7 illustrates an average of more than 90% correct responses across the whole cohort, and for individual year levels. The percentage of correct responses showed a consistent recognition of a fraction greater than one.

Table 5.7

Percentage of correct responses for Question 3: identifying the correct value of a fraction on a number line.

3. Which of the following numbers is the value of the point shown on the number line? (CIRCLE)



A. $\frac{3}{4}$ B. $\frac{7}{8}$ C. $1\frac{7}{8}$ D. $1\frac{3}{4}$

Year 7	95
Year 8	91
Year 9	92
Total	93

Among the incorrect responses were two errors found to be common to this type of question. Yanik, Helding, and Flores (2008) found that when a number line is marked to show numbers greater than 1, some students misapply part-whole relationships. Students in this study either omitted the whole number part or recognised that the number was greater than one but counted each check mark increment, therefore determining the answer had to be a fraction out of eight (Figure 5.2.). Multiple studies have reported the difficulties students experience when asked to locate a fraction on a number line (Bouchard, 1976; Byrnes & Wasik, 1991; Clarke, 2008; Creswell, 2009; Fazio & Siegler, 2010; Pearn & Stephens, 2007; Spangler, 2011; Vinner, 1997; Yin, 2009). Comparable studies found that naming fractions on a 0-to-2 number line is more difficult for students than naming fractions on a 0-to-1 number line, as students struggle to identify the unit. Novillis (1976) suggested a common misconception is that students view the total marked length of a number line as one whole, regardless of the values placed on it. When teaching fractions, it would be beneficial to make students really think about what constitutes the unit, and so a number line should at least go from zero to two.

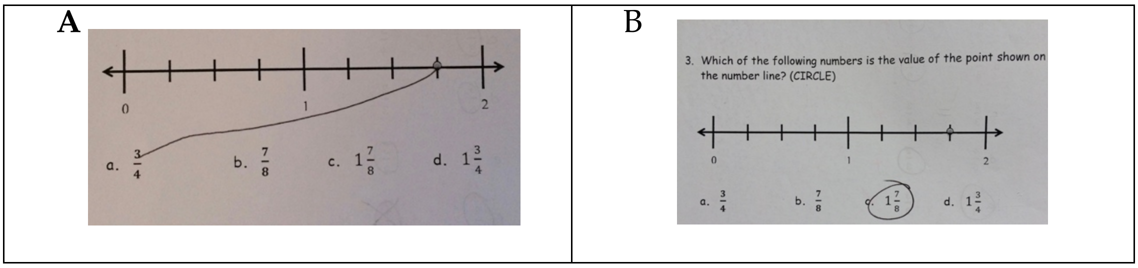


Figure 5.2. Illustrations of common errors when asked to name a fraction marked on a number line. Student omits the whole number portion (**A**) and student believes the unit is made up of eight parts but realises it is greater than one (B).

5.2.4 Place a set of given fractions on a number line

Supporting the findings of Ni (2000) that students often view the whole number line, irrespective of its magnitude, as a unit instead of a scale, Pearn and Stephens (2004) reported that students placed fractions with disregard to any other reference point or known fractions. Question 4 in this present study required students to place five different fractions on a number line. The questions varied among commonly used fractions, mixed fractions, negative fractions, and improper fractions (see Figure 5.3.). The graduations represented a whole number increment, with the number line ranging from -1 to 3. As can be seen in Table 5.8, percentage scores were lowest for the negative fraction and the improper fraction. Tolerance for a correct answer was based on the researcher's visual determination of what constituted "reasonable" precision.

Table 5.8

Percentage correct for Question 4 where students were asked to mark these fractions on a number line.

	a. $\frac{3}{4}$	b. $1\frac{1}{3}$	c. $-\frac{1}{4}$	d. $2\frac{1}{10}$	e. $\frac{5}{2}$
Year 7	82	89	71	87	64
Year 8	91	91	83	98	79
Year 9	87	87	76	93	69
Whole cohort	87	89	77	93	70

Figure 5.3 highlights some of the common misconceptions that were evident in the pre-test. Researchers have long argued that students' difficulty with analogue number lines stem from their inability to treat a fraction as a single entity instead of a two-number entity (Behr et al., 1993) and to recognise the density for rational numbers (Hiebert, Wearne, & Taber, 1991). In order for a fraction to have the attribute of size, it must be conceptualised as a single entity, even though it is composed as two whole-number symbols: the numerator and the denominator. To determine the size, students must be able to understand the relationship between the numerator and denominator. Students also need to understand that between any two rational numbers there is always another rational number (indeed, infinitely many) and must be able to judge the density for that relationship.

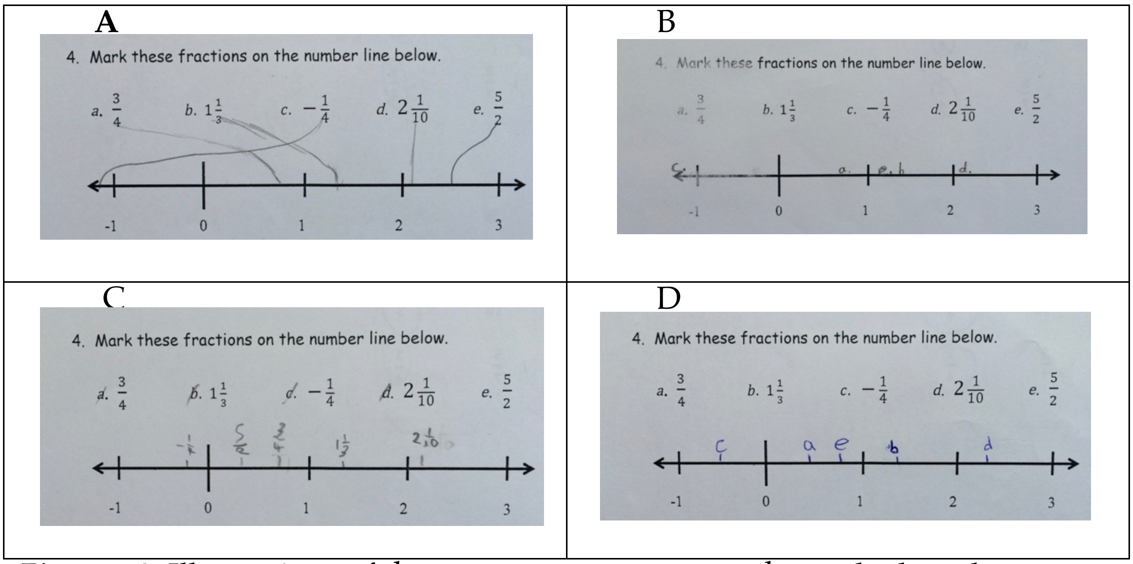


Figure 5.3. Illustrations of the most common errors when asked to place a series of fractions on a number line. Whole number bias, -1/4 is smaller than -1; improper fraction correct (**A**). Whole number bias, -1/4 is smaller than -1; improper fraction also incorrect (**B**). -1/4 correct; improper fraction incorrect (**C**). Improper fraction less than one; -1/4 incorrect but between 0 and -1 (**D**).

5.3 Part-Whole/Partition

Questions in this component of the pre-test related to fractions as operators and required students to shade a given fraction of a figure. These conceptual questions were designed to measure understanding of how much of a numerical quantity is represented by a fraction. Table 5.9 presents results for the questions examining understanding of fractions as operators.

Table 5.9

Percentage correct for Question 5: fractions as operators.

Question 5.	a	b	c	d	e	f	g
	a. $\frac{2}{3}$ *of* 9	b. $\frac{1}{3}$ *of* $\frac{1}{2}$	*c*. $\frac{7}{8}$ cake shared between 14 people	*d*. How many $\frac{1}{3}$ are there in $3\frac{1}{2}$?	Colour ¾ of this shape (rectangle)	Colour 2/3 of this shape (rectangle with 10 partitions)	Colour 2/7 of this shape (circle)
Year 7	78	49	45	56	93	47	67
Year 8	90	59	42	75	96	64	83
Year 9	87	50	43	59	97	52	76
Whole cohort	85	53	43	63	95	55	75

Item *c* proved to be the most difficult, with an average of only 43% of students answering this correctly. A common answer was $\frac{1}{2}$, as can be seen in Figure 5.4. Students calculated that when the cake was divided into 8 pieces, 7 pieces remained and each person would receive $\frac{1}{2}$ of one of those pieces. The calculations are correct but this is a transformation error where the original "whole" is not considered in final answer. Transformation errors occur when the student understands the needs of the question but fails to identify all the mathematical operations involved (Abdullah, Abidin, & Ali, 2015). In this case, the students have calculated the correct portion of the remaining pieces, however, have not considered this as a fraction of the original whole.

Item *b* was also poorly answered. Students made various procedural errors as well as having problematic conceptual representations. Answers included a correct answer through a conceptual diagram (Figure 5.4, "A"), showing the student's conceptual understanding of the question, but not illustrating whether this student would know how to answer this using an algorithm. Figure 5.4, "B" shows how the student incorrectly multiplied or kept the numerators and added the denominators. The student example in Figure 5.4 "C" shows how they convert the fractions to parts of 100 but leave the answer as a decimal rather than a percentage. This is also an example of a transformation error. This student has a conceptual understanding of the question but does not complete the problem.

Incorrect partitioning was the most common error for items *f* and *g*. Incorrect results in Figure 5.4 "J" show the student's lack of understanding of the importance of a standard referent unit.

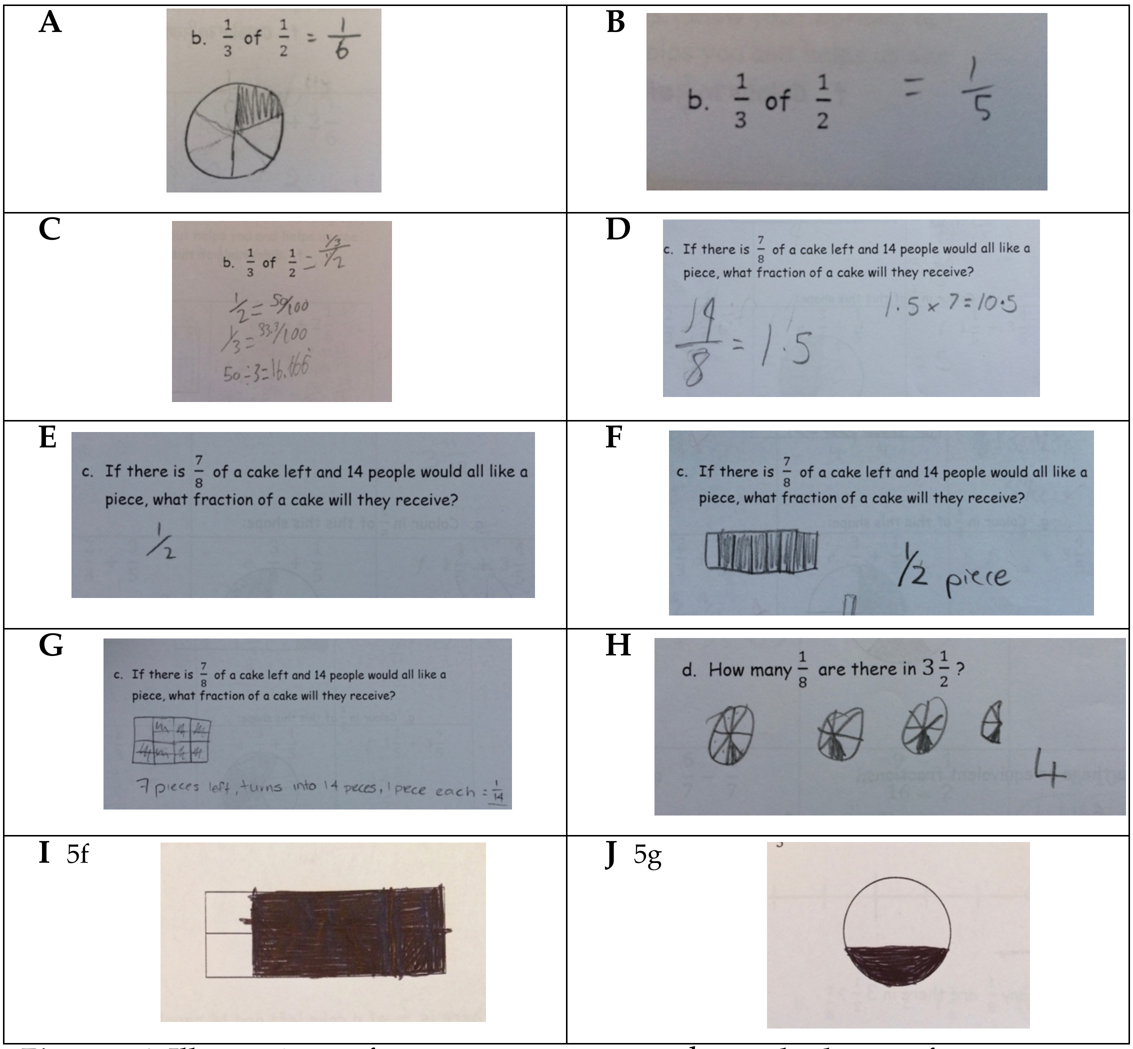


Figure 5.4. Illustrations of common responses when asked to perform part-whole operations.

Question 6 and Question 7 were items about fraction equivalence and simplification. The results of these two questions are discussed in section 5.4. Question 8 involved procedural/computational items and is discussed in section 5.5. In Question 9, students were asked to show the whole when $\frac{2}{3}$ of the shape was given. The percentage of correct responses for Question 9 are presented in Table 5.10.

Table 5.10.

Students were asked to show the whole when 2/3 of the shape was given.

Question 9. If this is 2/3 of a shape, draw a shape that shows the whole

Year 7	68
Year 8	86
Year 9	78
Whole cohort	77

It is difficult to determine if students who got this question wrong misunderstood the question, or if they did not know how to construct the whole given the portion. Figure 5.5, "A" and "B" show the most common error for Question 9 of the pre-test. Students have divided the shape into 3 parts and shaded 2 of the 3 parts. This error possibly reflects the fact that students are often asked to show a fraction of a whole, but not the reverse.

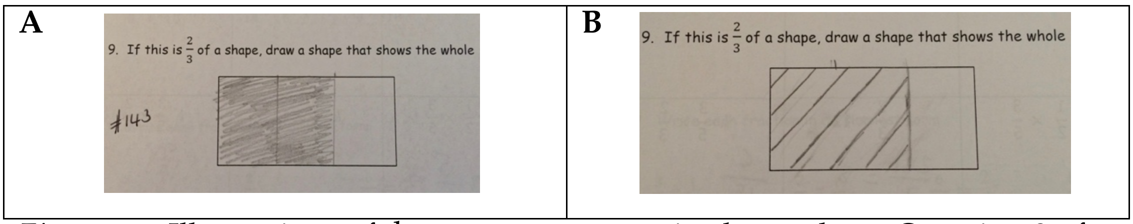


Figure 5.5. Illustrations of the most common mistake made on Question 9 of the pre-test.

Question 10 of the pre-test required students to interpret the parts of a collection model (see Figure 5.5 "A"). Table 5.11 presents the mean percentage of correct responses for each year level, as well as the percentage of correct responses across the whole cohort of participants. Students were not asked specifically to simplify the answer in part *a* of the question so it was surprising that so many students, particularly those in Year 9, got this question wrong.

Table 5.11.

Percentage of correct responses for Question 10.

Question 10. Parts of a collection model of fractions (see Figure 5.5 "A").		
	a. What fraction of dots is black?	b. What is another way of writing the same fraction?
Year 7	85	77
Year 8	90	84
Year 9	89	84
Whole cohort	88	81

When comparing the parts that are shaded to the total, the most common incorrect response was an answer of $\frac{3}{4}$ (see Figure 5.5 "A"). Although it was difficult to determine the exact reason behind this answer, part *b* of this question revealed some errors in forming equivalent fractions. In part *a*, students may have incorrectly determined that the equivalent of $\frac{12}{18}$ was $\frac{3}{4}$ (see Figure 5.5, "A").

Question 11 asked what fraction of the rectangle was shaded (see Figure 5.5, "B"). Table 5.12 presents the percentage of correct responses for this question.

Table 5.12.

Percentage of correct responses for Question 11.

Question 11. What fraction of the whole rectangle is shaded?	
Year 7	80
Year 8	85
Year 9	80
Whole cohort	81

In teaching fractions, we often show students wholes that are equally divided and assume students understand the importance of the parts being equal. In Figure 5.6, "D", the student has used the partitions to count 6 parts that have shading but has not considered that they are unequally sized. In example "C", the student has correctly calculated the numerator as equivalent to 4 parts but suggests these parts are out of 2, rather than the correct value of 6.

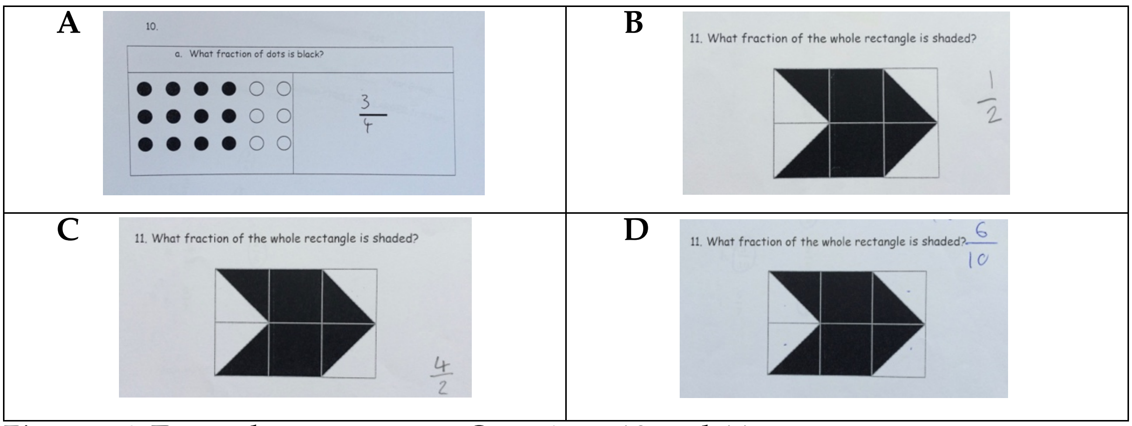


Figure 5.6. Example responses to Questions 10 and 11.

5.4 Equivalence and Simplification

Question 6 involved converting a series of three fractions to equivalent fractions. Table 5.13 presents the results as mean percentages of correct responses for Question 6.

Table 5.13

Percentage of correct responses for Question 6.

Question 6. Write these as equivalent fractions.			
	$a. \frac{3}{5} = \frac{}{10}$	$b. \frac{12}{18} = \frac{}{6}$	$c. \frac{1}{3} = \frac{}{18}$
Year 7	93	83	86
Year 8	96	89	89
Year 9	87	73	77
Whole cohort	92	82	84

The majority of students who did not receive a mark for each of these equivalent fractions questions did not attempt any parts of the question. It was difficult to determine misconceptions related to these equivalent fractions as no error patterns emerged. The Year 9 cohort had the lowest mean percentages for Question 6, much lower than their peers in both Years 7 and 8.

As can be seen in Table 5.14, the percentage of correct responses for Question 7 was dependent upon the fraction involved. More students were able to simplify $\frac{2}{8}$ than were able to simplify $\frac{3}{6}$.

Table 5.14

Percentage of correct responses for Question 7.

Question 7. Write each fraction in its simplest form.				
	$a. \frac{3}{6}$	$b. \frac{2}{8}$	$c. \frac{6}{15}$	$d. \frac{8}{1}$
Year 7	83	91	54	75
Year 8	90	96	75	88
Year 9	86	92	70	78
Whole cohort	86	93	66	80

For item *c* of Question 7, the majority of students who got this question wrong failed to find a common factor and left the fraction as $\frac{6}{15}$. Many students (12%) did not attempt this question at all. Nearly all of the incorrect answers for item *d* were a result of students' not attempting the question (61% of all incorrect responses).

5.5 Operations

Table 5.15 presents results from Question 8, which contained twenty-one procedural questions based on the operations of addition, subtraction, multiplication, and division. As detailed in section 4.5.5, the questions were designed to test understanding of variations of each operation, and included a mix of questions with like denominators, unlike denominators, and mixed fractions. The mixed fraction multiplication, item *o*, proved to be the most difficult, with a mean percentage of only 14% correct. Item *a*, addition with like denominators, was the most successful across the whole cohort, returning a mean percentage of 93% correct. The multiplication and division items had the lowest mean percentages of correct responses, in comparison to the results of the addition and subtraction items.

Table 5.15

Percentage of correct responses for Question 8: operations.

Question 8. Give all answers in simplest form.	a.	b.	c.	d.	e.	f.	g.	h.	i.	j.	k.	l.	m.	n.	o.	p.	q.	r.	s.	t.	u.
	$\frac{2}{\text{[illegible]}}+\frac{5}{\text{[illegible]}}$	$\frac{1}{2}+\frac{1}{4}$	$1\frac{3}{4}+2\frac{1}{6}$	$\frac{2}{3}+\frac{3}{5}$	$\frac{3}{4}+\frac{1}{5}$	$1\frac{1}{5}+3\frac{4}{5}$	$\frac{6}{7}-\frac{2}{7}$	$\frac{9}{16}-\frac{1}{2}$	$10-\frac{1}{3}$	$\frac{1}{2}\times\frac{3}{5}$	$\frac{2}{3}\times 12$	$\frac{3}{5}\times\frac{2}{3}$	$\frac{6}{7}\times\frac{14}{15}$	$6\times\frac{1}{3}$	$1\frac{3}{4}\times 2\frac{5}{6}$	$\frac{3}{5}\div\frac{1}{5}$	$\frac{1}{2}\div\frac{1}{4}$	$\frac{3}{4}\div\frac{1}{8}$	$\frac{1}{6}\div\frac{2}{3}$	$5\div\frac{1}{4}$	$4\div\frac{2}{3}$
Year 7	88	72	47	59	66	68	75	56	49	61	41	34	18	53	18	41	37	35	27	43	36
Year 8	94	86	68	78	82	81	92	69	63	79	58	37	20	57	16	53	59	49	41	49	43
Year 9	96	81	56	62	71	75	86	74	49	70	50	30	24	54	7	57	57	51	41	50	46
Whole cohort	93	80	57	66	73	75	84	66	53	70	50	34	20	55	14	50	51	45	36	47	42

In Figure 5.7, example "A", the student demonstrates the common misconception of adding the numerator and adding the denominator. This student applies this misconception regardless of the denominator being like or unlike, as seen in items *a, b* and *c.* The example "B" in Figure 5.7 shows that this student remembered to keep the denominator the same when adding with like denominators but when the denominators were different obtained the numerator and denominator from the sum. Example "C" in Figure 5.7 demonstrated a similar error to "B", in that this student subtracted the numerators and subtracted the denominators for both like and unlike denominators. This student did not recognise the error of having the undefined result of $\frac{4}{0}$. The example in Figure 5.7 "D" demonstrates the same error as in "B", where the student has answered the question with like denominators correctly but has subtracted both the numerators and denominators when the denominators were different.

In the multiplication questions, the errors varied and some examples are shown in Figure 5.7. Example "E" demonstrates how the student incorrectly inverted the second fraction when multiplying. In example "F", the student converted the fractions to a common denominator but multiplied numerators and left the common denominator as it was without multiplying. Example "G" shows that the student understood the correct procedure for multiplying fractions but did not know how to find the equivalent fraction of a whole number, for example in item *n* they have incorrectly replaced 6 with $\frac{6}{6}$. The student in example "H" did not convert the mixed fractions to improper fractions, or use the distributive law, to compute fraction multiplication of mixed fractions. The whole number parts and the fraction parts were treated separately, although the fractional parts were multiplied correctly.

In example "I", there was some recognition that fractions need to be converted to improper fractions but the student used the numerator of the first converted improper fraction as the numerator of the result and numerator of the conversion of second fraction as the denominator of the result.

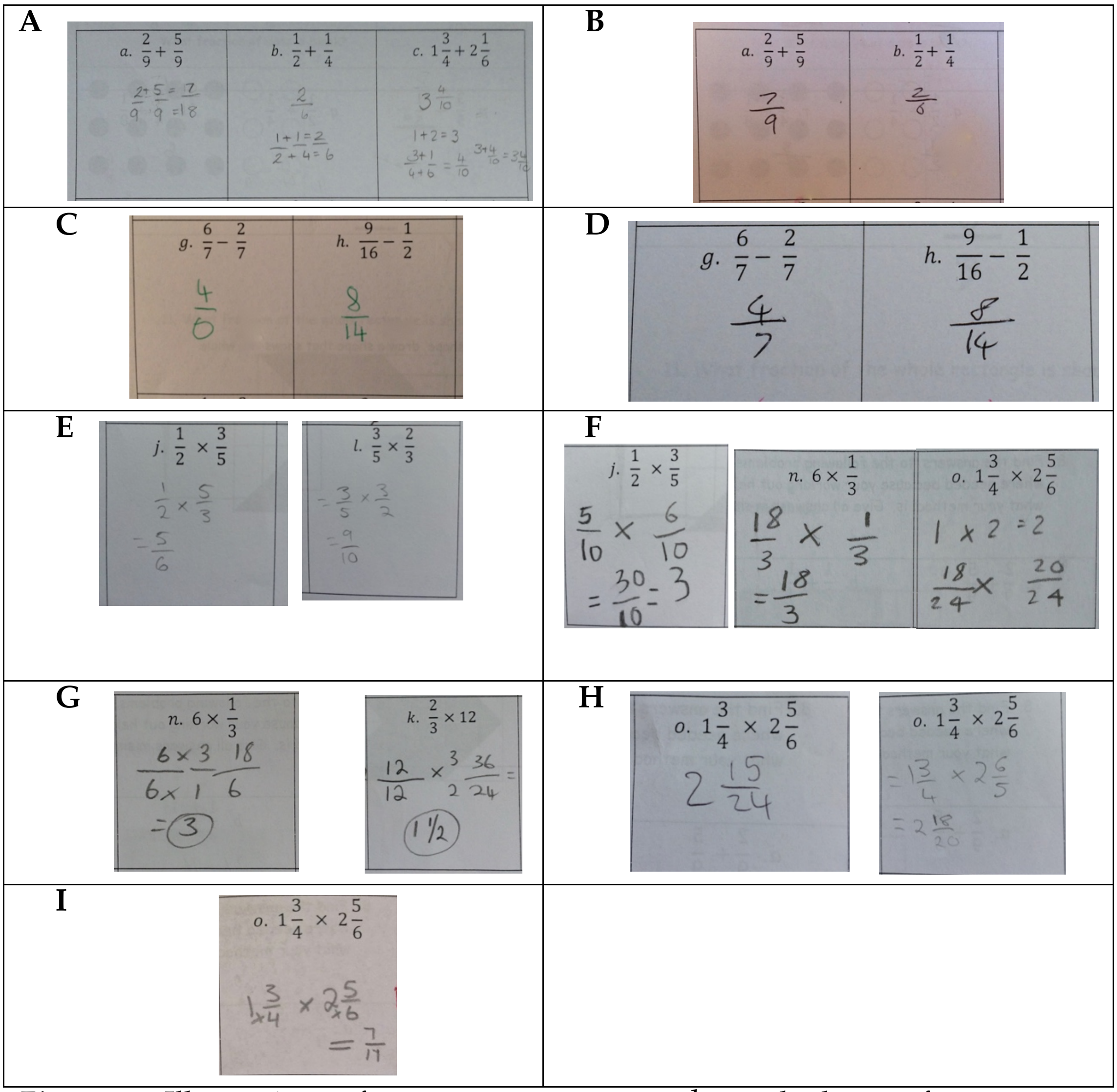


Figure 5.7. Illustrations of erroneous responses when asked to perform procedural operations.

In Figure 5.8 examples of the responses for the items in Question 8 that involved fraction division are shown. The student in example "A" appears to have inappropriately converted the fractions to have a common denominator and then subtracted the numerators ($\frac{3}{4} \div \frac{1}{8} = \frac{6}{8} - \frac{1}{8} = \frac{5}{8}$ and $\frac{1}{6} \div \frac{2}{3} = \frac{1}{6} - \frac{4}{6} = \frac{-3}{6}$). The student in example "B" has converted the fractions to have common denominators and then divided the numerators and denominators. This student has, however, incorrectly stated that the denominator divided by itself has the same value as the denominator instead of a value of 1 (as can be seen for items *p, q,* and *r*). In example "C" in Figure 5.8, the student has known that the numerator and denominators had to be multiplied but did not invert the second fraction first resulting in an incorrect answer for items *p, q,* and *r.* Example "D" shows a misconception that the denominator has to be divided by the numerator of each fraction, with the first fraction division (incorrectly evaluated as a whole number) forming the numerator of the result and the second fraction division forms the denominator part of the result.

In example "E", the student divides the numerators and divides the denominators but which fraction is divided by which appears to be determined by which one produces a whole number result. For example, in item *r* the student divides the first numerator by the second numerator ($3 \div 1$) but then divides the the second denominator by the first denominator ($8 \div 4$). Example "F", item *r,* illustrates an awareness that multiplication needs to be performed and cross-multiplication was used, however, this student incorrectly placed the answer for the resulting fraction around the wrong way. This example shows how the student multiplied the denominator of the first fraction by the numerator of the second fraction but wrote the answer of the resulting fraction as the numerator. This gave the reciprocal result. In item *p* of example "F", the student correctly inverted the second fraction but then cross multiplied, resulting in an incorrect answer.

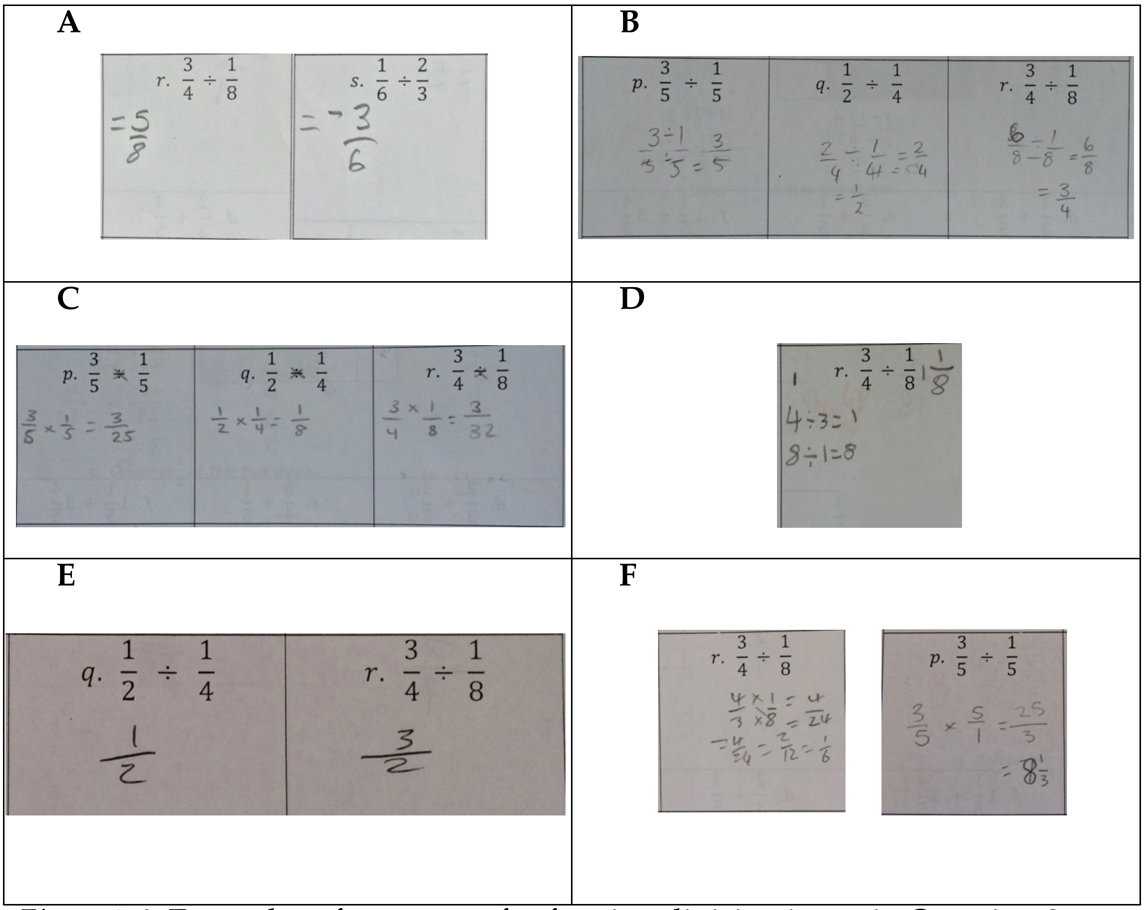


Figure 5.8. Examples of responses for fraction division items in Question 8

5.6 Summary

Conceptual understanding was examined through questions related to the part-whole, operator, quotient, and measure sub-constructs. The students in this study exhibited many common misconceptions in addition to some unique errors. Procedural understanding was determined with respect to the four operations of fractions: addition, subtraction, multiplication, and division. Procedural competence was also examined through questions of equivalence and simplification. Many of the misconceptions demonstrated in the student samples of the procedural questions in this study were common errors reported by previous research, however, some errors in procedure were unique to this study.

Question 8 was used to identify students with errors that might be able to be addressed via an intervention program. Students who had repeat errors in Question 8 were identified and invited to participate in one of the intervention programs. Results of the intervention programs are described in the next chapter, Chapter 6.

Chapter 6

Results: Intervention Programs

This chapter contains details and discussion about the student fractions test data analysis from each stage of the research, to examine the impacts of the intervention programs. The Fractions Diagnostic Test, as described in Section 4.5.5, was used to ascertain which students had misconceptions associated with fraction computation, as determined by repeated errors. Comparisons among the control group and the two experimental groups (O/N and Traditional Intervention) are presented, as well as comparisons examining the effectiveness of the intervention programs for the experimental groups. All resulting data was analysed by computer, using the SPSS version 24 program, with subsequent results being represented both in tabular and graphical format.

The following limitations were considered when interpreting the results:

- 83 students were identified as having misconceptions from the total of 361; however, many students did not attempt all questions and therefore did not present repeat errors.
- Of the 83 participants invited to take part in the intervention program, only 35 students completed the program from start to finish.
- Only intervention participants completed the post-test, so comparisons between the intervention groups after intervention and the control group were made on pre-test scores of the control group (five weeks earlier). The research model meant that it was not possible to additionally report on any significant changes in the scores of the control group until the delayed retention test.

Misconceptions and repeated errors were identified from students' procedural calculations in Question 8 of the Fractions Diagnostic Test. Tests of effectiveness of the intervention programs and comparisons to the control group are therefore reported based on the scores for Question 8 only. Moreover, it is important to note that the O/N intervention has procedural actions as a focus.

To examine the major research aim of determining the effectiveness of O/N method of intervention the following comparisons were made:

- Comparison of pre-test means of experimental and control groups,

- Comparison of means from pre-test to post-test for intervention participants,
- Comparison of the effectiveness of the intervention types,
- Comparison of delayed retention scores to both pre-test and post-test means for both control and intervention groups.

6.1 Comparison of Pre-Test Means

Descriptive statistics were used to highlight the differences between the experimental and control groups at the pre-test stage. The mean and standard deviation for Question 8 were computed separately for the students identified as having repeat error computational misconceptions and those who did not. The higher the mean score, the higher the achievement on Question 8 of the fractions test. Table 6.1 displays comparison results for the group with errors and the group with no errors, as determined by their results for Question 8. In this case, the "error group" consists of all students identified as having problems with the procedural questions (n=83), not only those identified as having problems and who also participated in the intervention (n=35). The Control group was the remaining cohort of students (n=278). Question 8 was out of a possible score of 21 and the total score was out of a possible 55.

Table 6.1

Descriptive statistics for the Pre-test scores for Experimental and Control Groups.

	Group	Mean	Std. Deviation	N
Q8 Pre-test	Control	12.84	5.91	278
	Q8 Error group	7.14	4.7	83
Total Pre-test	Control	41.39	10.67	278
	Q8 Error group	30.85	10.33	83

Between-subjects effects showed a significant difference between the Control and Experimental groups ($p < .001$) pre-test scores for Question 8. A significant difference ($p = .05$) was also found in Total Score results between the two groups. These results show the difference in the groups' fraction understanding at the pre-test stage. The standard deviation of the Question 8 scores highlight less variability in the Q8 Error group than for the control group. The aim of the intervention was to narrow the gap in fraction understanding between the experimental and control groups and to determine the effect of intervention, by measuring growth in fraction knowledge over time.

At the completion of the pre-test, students who were identified as having misconceptions in Question 8 were invited to participate in an intervention program. They were split into three groups; those with errors but who declined the invitation to participate in intervention (No Treatment group), an Old Way/New way Intervention Group, and a Traditional Intervention Group. The pre-test scores for these groupings of 'error' participants can be seen in Table 6.2 below.

Table 6.2

Descriptive statistics for the Pre-test scores for Error Participants.

	Group	Mean	Std. Deviation	N
Q8 Pre-test	No Treatment	6.65	4.21	48
	Old Way / New Way	7.75	5.70	20
	Traditional	7.40	4.69	15
	Total Q8 error	7.14	4.70	83

The mean values were calculated from a possible score of 21. All three groups of participants who were identified as having errors had similar mean scores, with the O/N intervention participants having slightly more variation in scores, as seen by the higher standard deviation. Unfortunately, the 48 participants who chose not to participate in the intervention program had the lowest mean value and, therefore, the least procedural fraction knowledge.

6.2 Comparison of Scores Pre-Post Test

Before any comparisons were made about the effectiveness of individual intervention programs, analysis was used to determine if the group mean of the combined intervention participants at the post-test stage were, at that stage, comparable to the pre-test mean of the non-intervention participants. First, Table 6.3 shows that intervention led to improved scores for the intervention participants; note the significant increase in mean score of the combined intervention participants after intervention.

Table 6.3

Comparison in mean scores pre-test to post-test for intervention participants.

Intervention Participants	Mean	Std. Deviation	N
Q8 Pre-test	7.38	5.13	35
Q8 Post-test	11.24	5.70	35

Preliminary analysis highlighted that intervention was an effective way of bringing error participants scores up to the 'average' range, as determined by pre-test scores. The pre-test mean score of the control group for Question 8 was 12.84, in comparison to the post-test mean of 11.24 for the intervention participants (see the top half of Table 6.4). Further investigation then determined the effectiveness of each of the two intervention programs and whether knowledge was able to be retained over an extended period of time.

Table 6.4

Comparison of pre-test score for non-intervention participants compared to post-test scores for intervention participants.

	Mean	Std. Deviation	N
Q8 Pre-test Non Intervention	12.84	5.91	278
Q8 Post-test Interventions combined	11.24	5.70	35
Q8 Post-test O/N	11.70	5.97	20
Q8 Post-test Traditional	10.87	5.30	15

Looking at the intervention groups separately, the second half of Table 6.4 show that the O/N intervention group had a higher post-test mean than the Traditional intervention group, suggesting this intervention is a more effective method. It should be noted, however, that the O/N group (M = 7.75) had a higher pre-test mean than the Traditional group (M = 7.40). The O/N group still had slightly higher pre-post gains, with a mean score gain of 3.95 compared to the Traditional group, which had a mean score gain of 3.47. Despite the difference in the mean score gain, this was not statistically significant.

The results confirm an increase in fraction knowledge after intervention for both intervention groups, with the O/N group displaying slightly more growth in fraction procedural knowledge at the post-test stage.

6.3 Effectiveness of the Interventions

6.3.1 Error Participants Question 8 Score Analysis

As previously stated, the students identified as having repeated errors in the pre-test were split into three groups: no treatment, O/N and Traditional. The students with errors who declined the invitation to participate in an intervention program were used as a “no treatment” comparison in determining both the short-term and long-term effectiveness of the intervention programs. These students were only tested at the pre-test and delayed retention test stages.

A two-way mixed ANOVA was used to determine whether there was a two-way interaction between the between-subjects and within-subjects factors of the intervention groups. There were no outliers in the data, as assessed by inspection of a boxplot for values greater than 1.5 box-lengths from the edge of the box. Test scores were normally distributed, as assessed by Kolmogorov-Smirnov's test ($p > .05$, see Table 6.5). Studentised residuals test scores were normally distributed, as assessed by Normal Q-Q Plot. There was homogeneity of variances, as assessed by Levene's test of homogeneity of variance ($p > .05$). There was violation of homogeneity of covariances, as assessed by Box's test of equality of covariance matrices. The results from the test of normality are summarised in Table 6.5.

Table 6.5

Test of Normality, assessed by Kolmogorov-Smirnov.

		Kolmogorov-Smirnov[a]		
	Treatment Group	Statistic	df	Sig.
Pre-test Q8 Score	No Treatment	.152	48	.007
	Old Way / New Way	.166	20	.150
	Traditional	.099	15	.200*
Post-Test Q8 Score	No Treatment	.150	48	.008
	Old Way / New Way	.174	20	.113
	Traditional	.123	15	.200*
Delayed Retention Q8 Score	No Treatment	.146	48	.012
	Old Way / New Way	.129	20	.200*
	Traditional	.170	15	.200*

*This is the lower bound of the true significance

Mauchly's test of sphericity indicated that the assumption of sphericity was violated for the two-way interaction, $X^2(2) = 12.361$, $p = .002$. Based on the recommendation by Maxwell and Delaney (2004), the results were interpreted using the Greenhouse-Geisser correction due to the extreme sensitivity of repeated measures ANOVAs to departures from sphericity.

Sphericity is an important assumption of repeated-measures ANOVA. It is the condition where the variances of the differences between all possible pairs of within-subject conditions are equal. The Greenhouse-Geisser is used to assess the change in a continuous outcome with three of more observations across time or within-subjects. The assumption of sphericity is violated for this type of within-subjects analysis and the Greenhouse-Geisser correction is robust to the violation.

Descriptive statistics in Table 6.6 highlight the changes in mean score for each of the treatment groups over time. As can be seen in the table, there is no score recorded for the "no treatment" group at the post-test stage. This is because this group only participated at the pre-test and delayed retention test stages. It was not within the scope of the study to re-assess their understanding 5 weeks after the pre-test as they had not participated in an intervention program. For the purpose of the research, it was assumed that their scores would be more or less the same at the post-test stage as they were at the pre-test stage especially since no fraction teaching took place in regular classes during that time.

The intervention groups had significant gains at the post-test stage for Question 8, although they were not able to maintain this at the delayed retention test stage the following year. Nevertheless, as Table 6.6 shows they did not revert to their original pre-test scores.

Table 6.6

Changes in Question 8 mean score for each of the three treatment groups over time.

	Treatment Group	Mean	SD	N
Pre-Test Q8 Score	No Treatment	6.65	4.21	48
	Old Way / New Way	7.75	5.69	20
	Traditional	7.40	4.69	15
Post-Test Q8 Score	No Treatment	----	----	48
	Old Way / New Way	11.70	5.98	20
	Traditional	10.87	5.30	15
Delayed Q8 Score	No Treatment	6.54	5.19	48
	Old Way / New Way	8.45	5.26	20
	Traditional	8.40	5.96	15

Figure 6.1 provides a graphical interpretation of the mean scores at the pre-test, post-test, and delayed retention test stages. The effect of intervention for the sample population is significantly different for the experimental groups, compared with the no treatment group. The O/N group had a marginally higher mean score at the post-test stage. Both intervention groups had gains at the delayed retention stage, but mean scores dropped again for the delayed retention test, suggesting intervention participants were unable to maintain their gains in mean score in the following school year, although they were higher than the original pre-test.

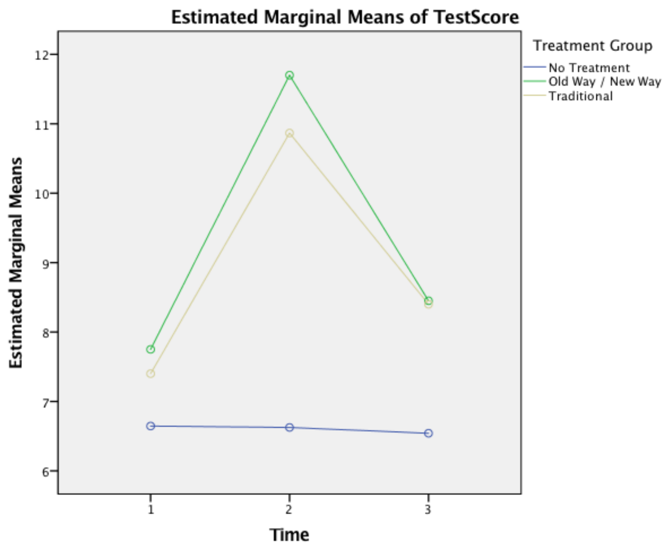


Figure 6.1 Plot of the Question 8 mean scores for the three treatment groups over time.

There was a statistically significant interaction between the intervention type and time of testing on the test score, $F(4, 160) = 4.44$, $p = .002$, $\eta^2 = .1$. There was a statistically significant difference in Question 8 test score between interventions immediately following the treatment phase at the post-test time point, $F(2, 80) = 9.594$, $p < .001$, partial $\eta^2 = .193$. This indicated that the intervention programs made a difference to the Question 8 post-test scores. The details of this difference were then explored further. In what follows, data are presented as mean ± standard error, unless otherwise stated. The test scores for Question 8 were statistically significantly different between the No Treatment Group and both the O/N Intervention Group (-5.08 ±1.3, $p = .001$) and the Traditional Intervention Group (-4.24 ± 1.44, $p = .012$). The Question 8 test scores were not statistically significantly different between the O/N Intervention Group and the Traditional Intervention Group (.83 ± 1.67, $p = .871$).

There was a statistically significant effect of time on Question 8 test score for the O/N Intervention Group, $F(2, 38) = 5.532$, $p = .008$, partial $\eta^2 = .226$. There was a statistically significant effect of time on Question 8 test score for the Traditional Intervention Group, $F(2, 28) = 6.393$, $p = .008$, partial $\eta^2 = .313$.

For the No Treatment Group, the test scores were not statistically significantly different between the two time points. This data shows that the students who received intervention significantly improved their fraction knowledge at the post-test stage compared to the no treatment group, who had no significant change in fraction knowledge. The two intervention groups both had gains but there was not a statistically significant difference between the two groups. They both improved markedly at the post-test stage but it not statistically conclusive if one intervention was more effective than the other.

For the O/N Intervention Group, the test score was statistically significantly different between the pre-test and the post-test ($M = -3.95$, $SE = 1.42$, $p = .036$), and the post-test and the delayed retention test (($M = -3.25$, $SE = .914$, $p = .006$). It was not statistically significantly different between the pre-test and delayed retention test ($M = -.700$, $SE = 1.40$, $p = 1.0$). For the Traditional Intervention Group, the test score was statistically significantly higher from the pre-test to the post-test ($M = -3.47$, $SE = .844$, $p = .003$). This data shows that although intervention was effective at the post-test stage for both groups, the gains in fraction knowledge were not able to be maintained at the delayed retention test stage the following year. As seen in the plot of mean scores in Figure 6.1, both intervention groups had higher mean scores at the delayed retention test stage in comparison to the pre-test score, however, this was not statistically significant. The no treatment group had no change in mean score at the post-test stage and a very minor decrease in mean score at the delayed retention test stage.

6.3.2 Error Participants Total Score Analysis

This section examines the Total score for each of the three treatment groups over time. There were three outliers in the data, as assessed by inspection of a boxplot for values greater than 1.5 box-lengths from the edge of the box. Case 82 was kept as it only represented a very low pre-test score. Cases 47 and 74 represented vast improvement after treatment and therefore remained in the analysis. Based on examination of the outliers, all three outliers were included in the analysis of the results. Test Scores were normally distributed, as assessed by Kolmogorov-Smirnov's test ($p > .05$). Test scores were normally distributed, as assessed by Normal Q-Q Plot. There was homogeneity of variances, as assessed by Levene's test of homogeneity of variance ($p > .05$). There was homogeneity of covariances, as assessed by Box's test of equality of covariance matrices ($p = .876$). Table 6.7 shows the descriptive statistics for the Total Score for each of the three treatment groups over time.

Table 6.7

Changes in Total mean score for each of the three treatment groups over time.

	Treatment Group	Mean	SD	N
Pre-Test TOTAL Score	No Treatment	30.00	10.17	48
	Old Way / New Way	32.85	9.44	20
	Traditional	29.87	11.87	15
Post-Test TOTAL Score	No Treatment	----	----	48
	Old Way / New Way	30.20	14.02	20
	Traditional	23.80	12.85	15
Delayed TOTAL Score	No Treatment	27.92	12.56	48
	Old Way / New Way	30.30	10.91	20
	Traditional	32.67	13.38	15

Figure 6.2 provides a graphical interpretation of the Total mean score at the pre-test, post-test and delayed retention test stages. There were no statistically significant changes in Total score for the O/N and no treatment groups. The Traditional intervention group had a significant increase in Total score between the post-test and delayed retention test stages. This was in contrast to the same group's Question 8 scores, where their mean score increased after intervention. Many intervention participants reported that they did not attempt all questions on the post-test as they were concentrating on Question 8 items as these were most relevant to the intervention programs. Forty percent of participants in the Traditional Intervention Group did not attempt Questions 1 through to Question 7 at all on the post-test, which would account for the decrease in total score at that stage. This difference is discussed in more detail at the end of this section.

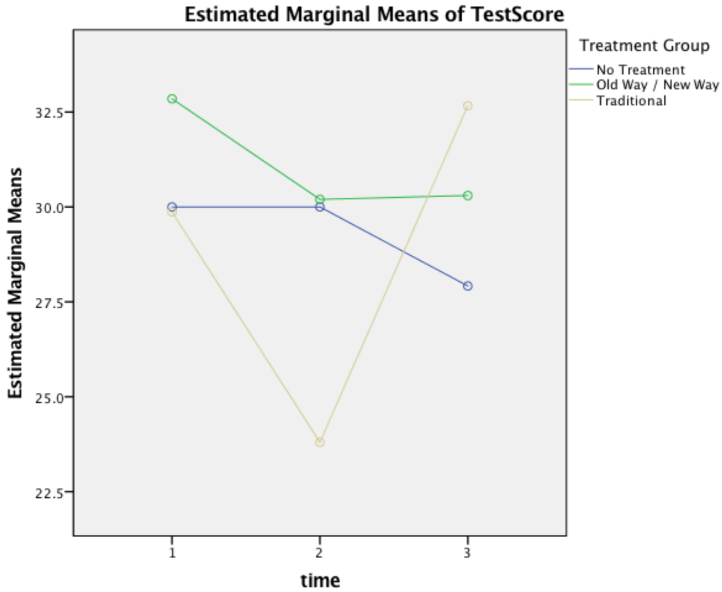


Figure 6.2 Plot of the Total mean scores for the three treatment groups over time.

Mauchly's test of sphericity indicated that the assumption of sphericity was violated for the two-way interaction, X2(2) = 18.267, p = .000. Based on the recommendation by Maxwell and Delaney (2004), the results were interpreted using the Greenhouse-Geisser correction due to the extreme sensitivity of repeated measures ANOVAs to departures from sphericity.

There was a statistically significant interaction between the intervention and time on the test score, $F(3.316, 132.621) = 4.037$, $p = .007$, $\eta^2 = .092$, $\epsilon = .829$. There was a statistically significant effect of time on total test score for the Traditional Intervention Group, $F(2, 28) = 5.256$, $p < .05$, partial $\eta^2 = .273$. For the Traditional Intervention Group, the total test score was not statistically significantly different between pre- and post-test points (M = 6.07, $SE = 3.04$, $p = .197$) or between pre-test and the delayed retention test (M = -2.80, $SE = 1.75$, $p = .395$), but it had statistically significantly improved between the post-test and the delayed retention test times (M = -8.87, SE = 3.34, $p < .05$).

6.3.3 Effectiveness of intervention on error type

A three-way mixed ANOVA was run to determine the effects of error type, intervention, and time on Question 8 test scores for reported misconception types. For this comparison, the misconceptions were divided into addition/subtractions errors, multiplication errors, and errors in division. Data are presented as "mean ± standard deviation" unless otherwise stated. Test scores were normally distributed, as assessed by Shapiro-Wilk's test ($p > .05$). and there were no outliers in the data, as assessed by inspection of a boxplot. There was homogeneity of variances for the post test scores and delayed retention score ($p > .05$), but not pre-test score ($p = .019$), as assessed by Levene's test for equality of variances. There was no three-way interaction between time, error type, and intervention type, $F(4, 58) = .792$, $p = .535$, partial $\eta^2 = .052$. This means that from the results it was unable to be established if the type of intervention had an effect on a specific type of error (addition, subtraction, multiplication or division).

Statistical significance of a simple main effect was accepted at a Bonferroni-adjusted alpha level of .025. There was a statistically significant simple main effect of the Old Way / New Way intervention group's pre-test score $F(2, 29) = 8.794$, $p < .001$, but not for the Traditional Intervention Group, $F(2, 29) = .100$, $p = .905$. On closer inspection, the O/N group had a greater number of addition/subtraction errors than either multiplication or division errors.

6.3.4 Comparison of intervention participants as one group and the effect of time

To determine the effectiveness of intervention on fraction understanding, comparisons were made between the control group pre-test and delayed retention test scores and the intervention group's score. The intervention groups were grouped together as one group for this comparison. A one-way repeated measures ANOVA was conducted to determine whether there were statistically significant differences in Question 8 scores over the course of a 5-week remedial intervention period. Delayed retention was also measured. There were no outliers and the data were normally distributed, as assessed by boxplot and Shapiro-Wilk's test ($p > .05$), respectively. The assumption of sphericity was not violated, as assessed by Mauchly's test of sphericity, $\chi^2(2) = 1.619$, $p = .445$. The remedial intervention elicited a statistically significant increase in Question 8 test scores ($M = 7.38$, $SD = 5.13$) to post intervention ($M = 11.24$, $SD = 5.7$). Post hoc analysis with a Bonferroni adjustment revealed that test score after intervention had a significant mean increase of 3.9, with 95% confidence interval [1.6, 6.1], $p < .001$.

As a comparison to the Intervention group scores, the Control group's scores were summarised. Table 6.8 displays results for the control group mean total score and Question 8 mean score for both the pre-test and the delayed retention test the following year. Although the results were not significantly different, the control group's score decreased over this time period.

Table 6.8

Descriptive statistics for the Control Group Pre-Test and Delayed Retention Test.

Control Group	Pre-Test			Delayed Post-Test		
	N	Mean	SD	N	Mean	SD
Total Score	278	41.39	10.67	278	35.09	15.85
Question 8 Score	278	12.84	5.91	278	10.19	6.51

Table 6.9 presents results for the two intervention groups (combined scores, regardless of intervention type). These results reflect the change in scores pre- to post-intervention and then again at the delayed test stage the following year. Both the Total score and Question 8 scores are reported on as the interventions were focused on the procedural errors found in Question 8 of the pre-test. The intervention groups had interesting results for their total test scores over time. Although none of these changes were statistically significant, the intervention group had a decrease in total score pre-post-test, with the mean score dropping from 30.85 to 27.06. The total score was then reported on again at the delayed retention test stage and the group increased from both their pre-test and post-test scores. Despite a decrease in the total score at the post-test stage straight after intervention, the group had a significant increase in their Question 8 score from pre- to post-test. The mean pre-test result (M = 7.38) increased significantly at the post-test stage (M = 11.24). The experimental group's Question 8 mean score decreased again at the delayed retention test stage but not significantly and it was still higher than at the pre-test level. Most important, were the results of the intervention group at the delayed retention test stage in comparison to the results of the

control group at the same stage. As can be seen in Table 6.9, the intervention groups score at the delayed retention test stage are comparable to those of the control group at the same stage. The intervention group mean was M=33.06, compared to that of the control group mean M=35.09.

Table 6.9

Descriptive statistics for the Intervention Groups: Pre-Test, Post-Test and Delayed Retention Test.

Intervention Group	Pre-Test			Post-Test			Delayed Post-Test		
	N	Mean	SD	N	Mean	SD	N	Mean	SD
Total Score	35	30.85	10.33	35	27.06	13.7	35	33.06	13.18
Question 8 Score	35	7.38	5.13	35	11.24	5.68	35	9.21	5.7

6.3.5 Results of observations, journal entries, and ad hoc interviews

The students from both intervention groups were attentive and interested throughout the lunchtime sessions. Students in the O/N group were given individual algorithms to solve depending on their error type. They solved the problem using the “old way” and then wrote about the method used. At the end of the session they wrote about what they had learnt. Initially the students wrote quite detailed accounts of what they were doing and what they learnt. As they became more familiar with the technique and had more practice solving error questions the entries in the journal focused less on understanding and more about their procedural errors perhaps suggesting they understood why they got the question wrong and then knew how to fix it. Examples for O/N included:

"I know I can't add the numerators together because of a rule but I cannot remember, so I'm trying different ways until I remember it". This student then wrote on a subsequent lesson, *"I found the LCD and converted the fractions to equivalent fractions using the LCD. I then only had to add the numerators and the denominator stayed the same, then I can simplify"*. This student stopped writing about what they were doing and just solved the problems after two sessions.

Students in the Traditional Intervention group wrote about what they were doing and what they had learnt throughout the sessions. They were given instructions as a class and were taught various techniques to solve fraction problems. They reported they were learning a lot from each session but their journal entries didn't reflect the same shift in understanding:

"To multiply a fraction by another fraction is the same as multiplying whole numbers, but in this case, it is with fractions". This student then wrote on a subsequent lesson, *"multiplying fractions, multiply across the top then multiply across the bottom"*.

6.4 Summary

The data that were used to determine the effectiveness of the intervention programs were collected using the results of the Fractions Diagnostic Test at three time points. The test was used to assess the relative change in score from the pre-test to the post-test (after intervention), and then again at the delayed retention test stage. Results were compared for both the control group and experimental groups at the post-test stage. The effectiveness of intervention type was measured at the post-test stage with results of the participants in the traditional intervention group compared to those participants in the O/N treatment group. Effectiveness of intervention type was also compared between the post-test and delayed retention test stages. General results of intervention (combined scores) were compared with results of the control group at the delayed retention test stage. Results highlighted the effectiveness of intervention, with Delayed Retention Test results of the Experimental Group total score comparable to the results of the Control Group total score.

Although the experimental group returned lower results for the total score after intervention, their Question 8 score improved significantly. Many students reported that they didn't attempt all of the other questions in the post-test as they were concentrating on the Question 8 procedural questions. This explanation may account for the increase in their total score result at the Delayed Retention Test stage, where they showed improvement from both their pre-test and post-test scores. It is also acknowledged that the results could have been limited or influenced by the students who had been identified as having repeat errors in the pre-test but who chose not to participate in intervention.

Chapter 7

Results: Self-efficacy

7.1 Introduction

This chapter presents and discusses the self-efficacy questionnaire data analysis from the pre-test and post-test stages of the research. The questionnaire was designed to measure the three domains of functioning: the affective, the cognitive, and the conative. Scores for each of the domains, as well as total self-efficacy scores from pre- and post-testing are discussed. Comparisons between the two intervention groups, as well as the extent of gender differences, are presented.

A main aim of the study was to examine the effect of the experimental intervention on students' mathematics self-efficacy. The following additional research questions were also explored:

Do students with lower achievement on the fractions diagnostic test have different self-efficacy beliefs in mathematics than the students with average and above average achievement?

Can participating in a remedial mathematics intervention program change self-efficacy beliefs?

Do the changes in self-efficacy beliefs differ depending on which intervention program the students participated in?

Do males and females have different mathematics self-efficacy beliefs?

As detailed in section 4.4.5, the adapted Self-Efficacy Questionnaire used in this study included 20 statements addressing the three psychological domains of functioning. The questionnaire included eight Affective Domain statements, six Cognitive Domain statements, and six Conative Domain statements. The numerical listing of statements for each domain did not necessarily correspond with the questionnaire item number; for example, Statement 4 in the Affective Domain was item 5 of the questionnaire. A total of six reverse scored questions were part of the questionnaire, as described in section 4.4.5. These are marked with an asterisk in Table 7.1 below. The students responded to the questionnaire using a 5-point scale, from Strongly Disagree to Strongly Agree.

Table 7.1

Statements for each domain from the Self-Efficacy Questionnaire

Affective Domain	**Statement**	**Item**
Statement 1:	Working hard leads to success in maths	1
Statement 2:	I look forward to my maths lessons	2
Statement 3:	Some people just cannot do maths*	3
Statement 4:	There is no point me trying in maths*	5
Statement 5:	I cannot change how good I am at maths*	7

Statement 6:	I often get a maths question wrong but do not understand why*	9
Statement 7:	I know if I am going to get a maths question right	10
Statement 8:	I enjoy doing fractions	11

Cognitive Domain	**Statement**	**Item**
Statement 1:	I feel that I can make a start on the problems I have to do in class	4
Statement 2:	I am interested in the things I learn in maths	6
Statement 3:	With fractions, I understand even the most difficult work	12
Statement 4:	I often worry that it will be difficult for me when working with fractions*	13
Statement 5:	I get tense when I have to do fractions homework*	15
Statement 6:	I like the challenge of a hard fractions problem	16

Conative Domain	**Statement**	**Item**
Statement 1:	When I really try I can get through most difficult tasks	8
Statement 2:	Even when a fraction problem looks hard, I know I can make progress with it.	14
Statement 3:	I find the teacher's help useful in maths class	17
Statement 4:	When I do better than usual in maths, many times it is because I tried a little bit harder	18
Statement 5:	If I make a mistake in maths, I try to find out where I went wrong	19

Statement 6:	I am the most powerful influence on my own achievement in maths	20

* items that were reversed scored

7.2 Reliability

The 5-point-scale questionnaire consisted of 20 questions. Overall the scale had a high level of internal consistency for the pre-test score, as determined by Cronbach's alpha ($\alpha = .85$). The subscales also had adequate reliability.

Table 7.2 reports the 361 students' total mean score, standard deviation, and Cronbach's alpha for the Self-efficacy Questionnaire. Anxiety items are reverse scored, and there were 6 such items in total. These included: item *3*, "some people just cannot do maths", item *5*, "there is no point in me trying in maths", item *7* "I cannot change how good I am at maths", item *9*, "I often get a maths question wrong but do not understand why, item *13*, "I often worry it will be difficult for me when working with fractions", item *15*, "I get tense when I have to do fractions homework". All but the last two items are associated with the affective domain. If a student strongly agreed with this statement, their score of 5 would be reversed scored to 1 in the results, to indicate low self-efficacy.

Table 7.2

Students' descriptive statistics for the Self-Efficacy Questionnaire pre-test.

Pre-test	Mean	SD	Cronbach's Alpha
Affective Domain	3.69	.442	.65
Cognitive Domain	3.37	.682	.79
Conative Domain	4.05	.460	.63
Overall	3.70	.453	.85

The next section discusses the results in the context of each of the three domains.

7.3 Findings from the Self-Efficacy Pre-Test

7.3.1 Statistical Methods

Descriptive statistics were used to report the students' responses from each of the 20 items and comparisons between the intervention participants and the control group were conducted. For pairwise comparisons between the intervention and control groups t-tests were used, with all available data used in the independent t-tests. For the paired (repeated measures) t-tests only responses from the intervention participants were used as these cases were the only ones in which both pre- and post-test scores were recorded. The items were also divided into each of the three domains, Affective, Cognitive and Conative for comparison.

Heiberger and Robbins (2014) recommended the use of diverging stacked bar charts as the primary graphical presentation technique for Likert scales. Diverging stacked bar charts position replies from Likert scale responses horizontally, so that positive responses are stacked to the left of a vertical baseline and negative responses are stacked to the right of this baseline. Proportions of responses at different agreement levels are in percentages and are coded light (for closer to neutral) to dark (for more distant from neutral). The segment of the graph representing percentages for respondents who neither agree or disagree are split down the middle and are shown in a neutral colour. The bars are of equal length and depict the proportion of responses (percentages) at each Likert level (see, for example, Figure 7.1). Each full bar is the same length and depicts the 100% of responses.

7.3.2 Pre-test Affective Domain Responses

The affective statements asked the participants to consider their beliefs and examined their perceived status as a learner. Figure 7.1 shows the percentage results of the eight affective statements with the mean values calculated.

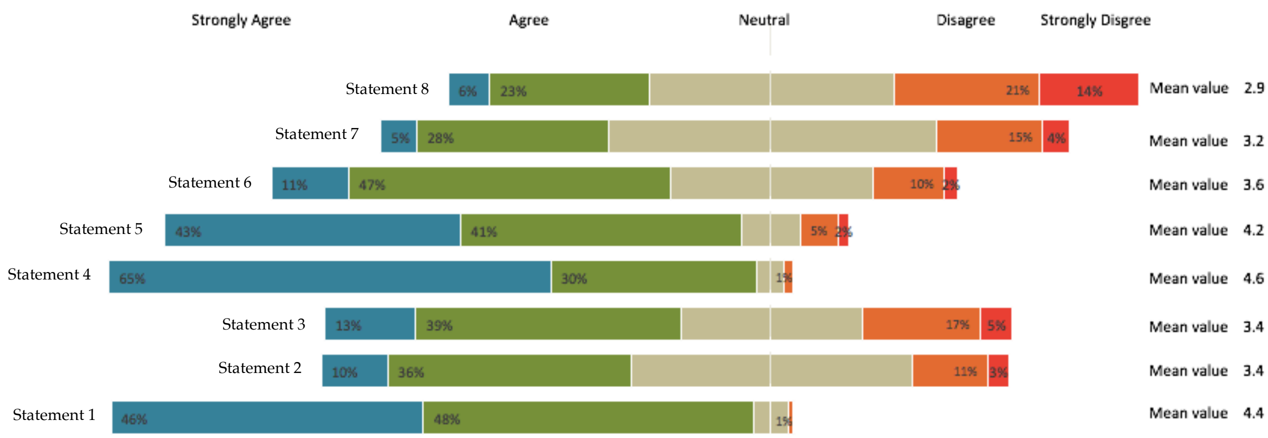


Figure 7.1. Participants' (n=361) affective domain questionnaire results.

There was strong agreement that working hard leads to success in mathematics (Statement 1, 94% agree or strongly agree), with 95% of the students disagreeing that there is no point in them trying and 84% disagreeing that they cannot change how good they are at mathematics (Statement 4 and 5 respectively, negative scores reversed). This suggests that the students believe that success can be attributed to hard work and that anyone can improve. This belief aligns with the work of Dweck (2006) and the notion of the "growth mindset". Despite this belief that it is worthwhile to strive in mathematics, there was only just a majority (52%) of the students disagreeing that "Some people just can't do maths", and a surprising 22% agreed with this statement. This supported the findings of Tanner and Jones (2003) who reported 93% of students thought it was worthwhile to try hard and a "worrying hard core" (p. 279) of 28% of the students who agreed that some people just cannot do maths.

Fifty-eight percent of students disagreed with the statement that they did not understand why they got a question wrong, whereas only 32% had the confidence to claim in advance that they know when they are going to get a question right (Statements 6 and 7 respectively).

7.3.3 Pre-test Cognitive Domain Responses

The cognitive domain involves the students' awareness of their mathematical knowledge, their strengths and weaknesses, and their ability to make connections with the curriculum (Tait-McCutcheon, 2008). Figure 7.2 shows the agreement/disagreement percentages and mean values of the six statements in this domain.

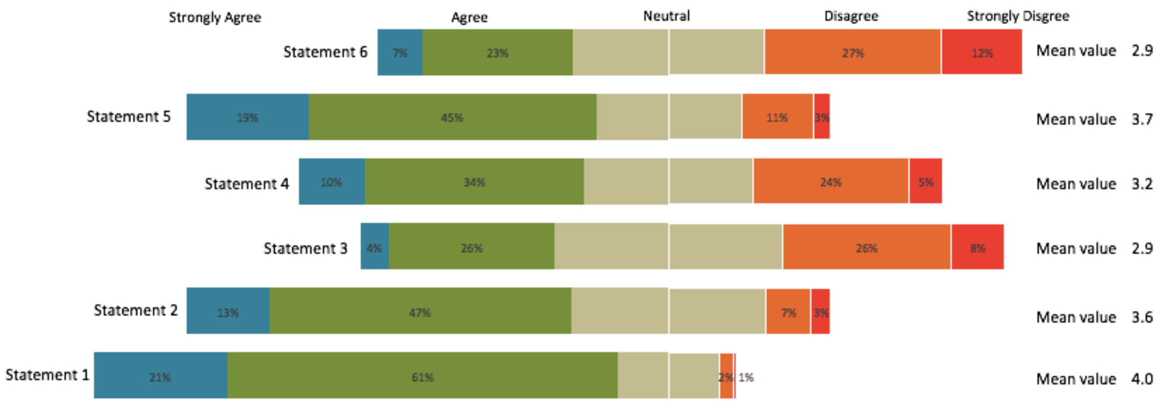


Figure 7.2. Participants' (n=361) cognitive domain questionnaire results.

The students were confident in their ability to make a start on problems in class (Statement 1, 82% agree and strongly agree) and expressed a general interest in things they learn in mathematics (Statement 2, 60%). When questioned specifically about their feelings towards fractions work, only 30% agreed/strongly agreed that they understood difficult fraction work (Statement 3) and, similarly, only 30% agreed that they liked the challenge of hard fractions problems (Statement 6). Questions related to the anxiety evoked from doing fractions suggested 29% worried the work would be difficult for them, with only 44% disagreeing with this statement (Statement 4). Sixty-four percent of students disagreed that they get tense when having to do fractions homework. This was much lower than the 44% of responses disagreeing with Statement 4, "There is no point me trying in maths". This may suggest the students were more anxious about fractions in classwork than they were about homework tasks.

7.3.4 Pre-test Conative Domain Responses

Huitt and Cain (2005) defined conation as the mental process that activates and/or directs behaviour and action. Statements in the conative domain asked students to provide responses about their motivation to learn mathematics, their self-direction, and their self-regulation. Figure 7.3 shows the percentage and mean values of the six statements in this domain.

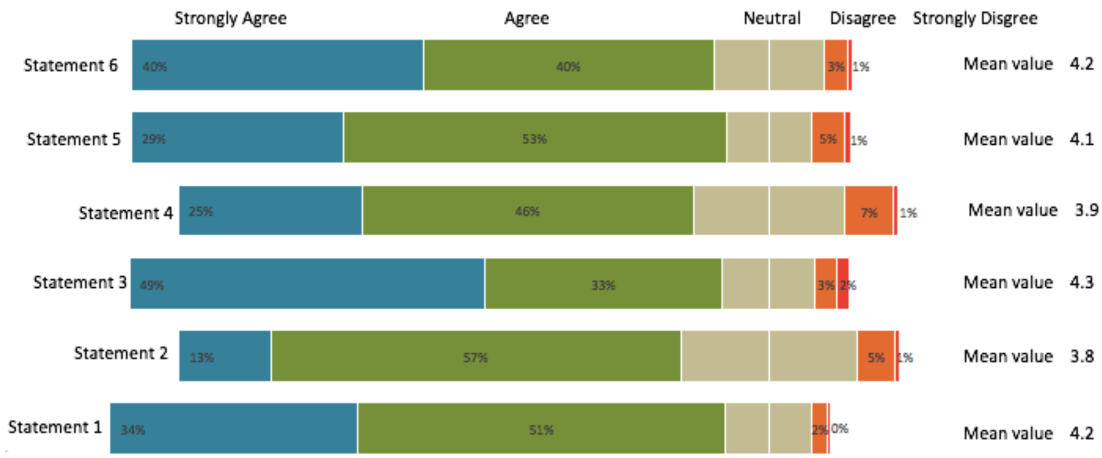


Figure 7.3. Participants (n=361) conative domain questionnaire results.

The conative domain responses returned the highest mean value (*M* = 4.05) with a high percentage of students attributing their success to effort. Eighty-five percent agreed they could get through difficult tasks when they really tried (Statement 1) and 71% reported that they did better than usual because they tried a little bit harder (Statement 4). There was a strong consensus that they, personally, were the most powerful influence on their achievement in maths (Statement 6, 80%), with 82% also agreeing that they found the teachers help useful in mathematics. The students agreed it was important to find out where they went wrong (Statement 5, 82%) and 70% agreed they could make progress with problems that looked difficult.

7.3.5 Comparison of "Error" with "Non-Error" Participants

Responses from the Self-Efficacy questionnaire were also examined by group: those participants who were identified as having repeated errors on the fractions diagnosis test, and those who did not (as outlined in section 4.5.2). The data analysis investigated the additional research question, "Do students with lower achievement on the fractions diagnostic test have different self-efficacy beliefs in mathematics than the students with average and above average performance?". "Error" was the de facto criteria for "low achievement" for this study.

The means and the standard deviations of the distributions were computed separately for the students in each of the two groups. There were no outliers in the data, as assessed by inspection of a boxplot. The Self-efficacy score was normally distributed, as assessed by Shapiro-Wilk's test ($p > .05$).

There were 81 students with repeated errors and 273 participants with no repeated errors on the fractions test. The students in the no repeated errors group from the fractions pretest ($M = 3.75$, $SD = 0.42$) had a higher self-efficacy score than those identified as having repeated errors ($M = 3.60$, $SD = 0.41$). There was homogeneity of variances for self-efficacy scores for error and no error participants, as assessed by Levene's test for equality of variances ($p = .748$).

An independent-samples t-test was conducted to compare the significance of the difference between the mean of the 'no errors' group with the mean of the 'errors' group. The 'no errors identified' self-efficacy score was 0.12 (95% CI, 0.22 to 0.02) higher than 'errors identified' self-efficacy score. There was a statistically significant difference in mean self-efficacy scores between participants with no errors identified at the pre-test and those identified with errors, [t(352) = -2.293, p = .022, d = 0.42].

These results were consistent with the suggestion by Tanner and Jones (2003) that students who attribute success or failure in mathematics to uncontrollable factors are unlikely to apply effective learning strategies. As Wolters and Rosenthal (2000) suggested, students with higher levels of self-efficacy set higher goals, apply more effort, and persist longer in the face of difficulty.

7.4 Comparison among intervention participants

The means and standard deviations of the distributions were computed for the intervention group as a whole, and separately for the O/N intervention group and the Traditional intervention group at the pre-test and post-test stages. The self-efficacy scale was composed of three domains, and the components of each of these domains was also examined. Descriptive statistics comparing the pre- and post-test results are presented in Table 7.3.

Table 7.3

Mean and Standard Deviation for Self-efficacy Questionnaire pre- and post-test

	Pre-Test			Post-Test		
	N	Mean	SD	N	Mean	SD
Combined	35	3.60	.456	35	3.62	.394
O/N	20	3.56	.488	20	3.68	.468
Traditional	15	3.76	.282	15	3.77	.395

There were no outliers in the data, as assessed by inspection of a boxplot. The differences between the self-efficacy score pre- and post-test were normally distributed, as assessed by Shapiro-Wilk's test (p = .486). Paired samples t-tests were used to compare mean values pre- and post-intervention. There was no significant change in the total self-efficacy score after intervention for the O/N and Traditional groups combined, $t(34) = .752$, $p = .458$.

An independent samples t-test was conducted to compare the significance of the difference between the mean of the O/N group with the mean of the Traditional group (Pre-Test). The Traditional group's mean self-efficacy score was 0.20 (95% CI, -.381 to .148) higher than the O/N group's mean self-efficacy score. There was no statistically significant difference in the mean scores between the groups at the pre-test stage ($t = .837$ for 34 degrees of freedom and $p = .409$).

At the post-test (Table 7.3) the mean of the traditional group was still higher than the O/N group, with both values increasing marginally: the traditional group's mean increased from 3.76 to 3.77 and the O/N groups mean increased from 3.56 to 3.68. An independent samples t-test was conducted to compare the significance of the difference between the mean of the O/N group with the mean of the Traditional group (Post-Test). There was no statistically significant difference in the mean scores between the groups at the post-test stage ($t = -1.375$ for 34 degrees of freedom and $p = .179$).

A paired samples (repeated measures) t-test was conducted to evaluate the significance of the changes between the pre-test and post-test stages for O/N and Traditional groups separately. Although the mean of the

O/N group increased, the increase was not statistically significant, $t(19) = .955$, $p = .352$. There was no statistically significant change in the Traditional group's mean, $t(14) = .057$, $p = .955$.

An examination of the three domain components that make up the self-efficacy scale will be presented in the following sections.

7.4.1 Affective Domain

The means and the standard deviations of the distributions for the affective domain were computed separately for the students in the O/N and Traditional Intervention groups. The results for the affective domain are shown in Table 7.4. The higher the mean score, the stronger the student's internal belief system.

Table 7.4

Mean and Standard Deviation statistics for the affective domain, pre- and post-test.

	Pre-Test			Post-Test		
	N	Mean	SD	N	Mean	SD
Combined	35	3.67	.469	35	3.69	.405
O/N	20	3.54	.553	20	3.68	.468
Traditional	15	3.84	.252	15	3.82	.362

The mean of the Traditional group at the pre-test stage was greater than the mean of the O/N group, indicating that before intervention the traditional group reported a greater belief in their potential to succeed than the O/N group. An independent samples t-test was conducted to compare the significance of the difference between the mean of the Traditional group with the mean of the O/N group at the pre-test stage, and a statistically significant difference was found, $t(33) = -2.18$, $p = .038$.

At the post-test stage, although the mean of the traditional group remained higher than the O/N group, there was no change in the traditional group's mean and the O/N group increased from 3.54 to 3.68. An independent samples t-test was conducted to compare significance of the difference of the means at the post-test stage. No statistically significant difference was found, $t(30) = -1.70$, $p = .099$.

To evaluate the significance of the differences between the pre-test and the post-test stages for the O/N and traditional group separately, a paired samples t-test was conducted. Although the mean of the O/N group increased, the change was not statistically significant, $t(18) = .112$, $p = .912$. There was no statistically significant change for the traditional group, $t(13) = -.071$, $p = .944$.

7.4.2 Cognitive Domain

The mean and the standard deviation of the distribution for the 6 items for the cognitive domain were computed separately for the students in the O/N and Traditional Intervention groups. The results for the cognitive domain are shown in Table 7.5. The higher the mean score, the greater the students' awareness of their own mathematical knowledge.

Table 7.5

Mean and Standard Deviation statistics for cognitive domain, pre- and post-test.

	Pre-Test			Post-Test		
	N	Mean	SD	N	Mean	SD
Combined	35	3.31	.595	35	3.21	.585
O/N	20	3.32	.621	20	3.12	.633
Traditional	15	3.31	.580	15	3.32	.517

The mean of the traditional group at the pre-test stage was the same as the mean of the O/N group, indicating that before intervention the groups reported similar awareness of their own mathematical knowledge. An independent samples t-test was conducted to compare the significance of the difference between the mean of the traditional group with the mean of the O/N group at the pre-test stage. There was no statistically significant difference, $t(33) = .027$, $p = .979$.

At the post-test stage, although the mean of the traditional group was higher than the O/N group, there was no change in the traditional group's mean and the O/N group decreased from 3.32 to 3.12. An independent samples t-test was conducted to compare significance of the difference of the means at the post-test stage. No statistically significant difference was found, $t(33) = -.986$, $p = .331$.

To evaluate the significance of the differences between the pre-test and the post-test stages for the O/N and traditional group separately, a paired samples t-test was conducted. Although the mean of the O/N group decreased, the change was not statistically significant, $t(18) = 1.13$, $p = .274$ There was no statistically significant change for the traditional group, $t(14) = .065$, $p = .949$.

7.4.3 Conative Domain

The mean and the standard deviation of the distribution for the 6 items of the cognitive domain component of the self-efficacy questionnaire were computed separately for the students in the O/N and Traditional Intervention groups. The results for the cognitive domain are shown in Table 7.6. The higher the mean score is, the greater the students' dispositions to strive to learn.

Table 7.6

Mean and Standard Deviation statistics for conative domain, pre- and post-test.

	Pre-Test			Post-Test		
	N	Mean	SD	N	Mean	SD
Combined	35	3.84	.456	35	4.12	.558
O/N	20	3.62	.439	20	3.98	.619
Traditional	15	4.07	.441	15	4.33	.464

The mean of the traditional group at the pre-test stage was higher than the mean of the O/N group, indicating that before intervention the traditional group reported greater volition. An independent samples t-test was conducted to compare the significance of the difference between the mean of the traditional group with the mean of the O/N group at the pre-test stage. There was a statistically significant difference, $t(33) = 2.995$, $p = .005$.

At the post-test stage, although the mean of the traditional group was still greater than that of the O/N group, both group's means increased, with the O/N group reporting a greater change. An independent samples t-test was conducted to compare significance of the difference of the means at the post-test stage. No statistically significant difference was found, $t(33) = 1.834$, $p = .076$.

To evaluate the significance of the differences between the pre-test and the post-test stages for the O/N and traditional group separately, a paired samples t-test was conducted. There was a statistically significant change in the O/N group mean between pre and post-test, $t(18) = 5.551$, $p < .000$. There was also a statistically significant change for the traditional group, $t(14) = 5.351$, $p < .000$.

7.4.4 Results of observations, journal entries, and ad hoc interviews

Students from both intervention groups made journal entries after each session. Students from the O/N group reported:

"I'm really confident with fractions now, this has really helped"

"I feel like I have learnt a lot more than when I first started"

"This gives me more confidence to try a problem"

"I feel like I am better at fractions now"

Students from the Traditional Intervention group also reported positive outcomes of the intervention:

"I feel like a have a better understanding of fractions now"

"I enjoy going to the fractions sessions"

"I have learnt things I did not know before"

Students in the O/N group reported on having more confidence and skills to solve problems, whereas the Traditional group reported on a greater understanding of fraction knowledge.

7.5 Gender Differences

A number of statistical tests were conducted to determine differences between males and females' self-efficacy pre- and post- intervention. A summary of the findings is detailed in this section.

There were no significant differences in male and female self-efficacy scores at the pre-test stage. There was no difference between males and females with repeated errors. There was no difference between males and females with no repeated errors. No differences were found between males with repeated errors compared with males who were found to have no repeated errors. At the pre-test stage there was a significant difference between females with repeated errors and those with no repeated errors. Females with errors had a lower mean score (M = 3.58) compared to those females with no repeated errors (M = 3.77). Investigating this difference more closely it was found that the significant difference occurred in the affective domain score and no difference was found in the cognitive or conative domain scores. This suggests that females who had repeated errors reported a perceived lower status as a learner of mathematics and reported that they believed they had a lower capacity to learn and less potential to succeed in comparison to the females who were not found to have repeated errors on the fractions diagnostic test.

A three-way mixed ANOVA was used to determine relationships among the two intervention groups (O/N and Traditional), gender, and time. There was no statistically significant interaction between gender and time of testing on self-efficacy, $F(1, 32) = .164$, $p = .688$, partial $\eta^2 = .005$. The main effect of group did, however, show a statistically significant difference in mean self-efficacy scores between gender groups $F(1, 32) = 3.411$, $p = .044$, partial $\eta^2 = .196$.

7.6 Summary

Traditionally, there has been a belief that students first like a task or topic and are then drawn to the activity due to their personal interest in the topic. As they engage with the activity over time, they develop expertise, knowledge, and skills, and from the development of expertise, their self-efficacy beliefs develop (Renninger, Hidi, & Krapp, 1992). The interest-first perspective is a strong belief held by many teachers and they often worry how to interest students in content, and they see interest as a prerequisite to all learning and future motivation. In contrast to this belief, however, Eccles et al. (1998) found there was an alternative path to motivation and learning, and they suggested students' interest and value beliefs might develop out of judgments of competence.

Bandura (1977) believed that the development of life-long learners of mathematics depended on the interaction and correlation of the three linked psychological domains of functioning: the affective, the cognitive, and the conative. That is, if students are cognitively and motivationally engaged, they are likely to be behaviourally engaged. In comparison to self-concept, which reflects more general beliefs about competence, self-efficacy beliefs refer to much more specific and situational judgments of capabilities.

General findings of this research suggested that the students believed that success is attributed to hard work and anyone can improve. Twenty-two percent agreed that some people just cannot do maths. Fifty-eight percent disagreed they did not understand why they got a question wrong, but only 32% had the confidence to claim they knew when they were going to get a question right. The participants in the pre-test returned a mean total self-efficacy score of 3.70, the conative domain was the highest ($M = 4.05$), and the cognitive domain was the lowest ($M = 3.37$). Students in the "no error" group had a significantly higher mean self-efficacy score (M = 3.75) than those "error" group participants ($M = 3.60$).

Comparisons of self-efficacy was made between the two intervention groups. The traditional intervention group had a significantly higher affective mean score at the pre-test stage. There was no significant difference in score at the post-test stage, suggesting that the O/N had a greater improvement in affective domain after intervention. The traditional group also had a significantly higher conative domain score at the pre-test stage. Both groups reported significant increases in conative domain mean scores at the post-test stage and as there was no significant difference between the two groups after intervention, this highlights a greater improvement in score by the O/N group. Intervention had a positive effect on both groups' conation, suggesting that intervention enabled the students to improve their disposition to strive to learn, and gave them more inclination to plan, monitor, and evaluate their work. This supports Graham and Weiner's (1996) findings that high self-efficacy and improved performance result when children adopt short-term goals, are taught to use specific learning strategies, and receive performance-contingent rewards. This finding also supports Dweck's (2006) work about the importance of a growth mindset. When students believe that everybody's ability can grow, their achievement improves significantly.

It is unknown, and beyond the scope of this study, how positive psychology traits such as grit and mindset may have influenced the results beyond the intervention. A study by Kahn (2018) found that students' level of fixed mindset was a strong predictor of achievement in mathematics, suggesting this factor should be considered when interpreting the results.

Chapter 8

Conclusion

8.1 Introduction

This thesis has focused on students' fraction understanding and the effects of remedial instruction on understanding, computational skills, and self-efficacy. One aim of the study was to elicit, and make visible, fraction computation misconceptions of secondary students, supporting the literature that achieving a depth of understanding in fractions is both complex and difficult. It was found that students do not construct meaning in isolation; rather, they try to make sense of new ideas based on what they already know. It was found that students often do not remember which procedural processes to use when doing fraction computation and this, coupled with a lack of deep understanding, means they often do not experience success with fractions. As a result, students become despondent about their ability and achievement in the topic, leading to low self-efficacy. This final chapter describes the outcomes of this study and uses these outcomes as a basis for the formulation of a number of conclusions and recommendations. An overview of the chapters is provided next.

8.2 Summary

Chapter 1 (Introduction) described the background and rationale for the study. Highlighted in Chapter 1 was the fact that although errors are a significant part of the learning process, there are instances when students connect new information with pre-conceived knowledge, where either the preconceptions are inappropriate or the new connections are not correctly constructed. It was discussed that prior knowledge may not always be correct knowledge; therefore, it is important that the preconceptions of students are assessed and dealt with when found to be incorrect. It was suggested that unless interventions are conducted by teachers these errors will persist.

The need to investigate how teachers can enhance the fraction understanding of students and eradicate misconceptions was discussed. The research presented in this thesis addressed the question of how an intervention program, which focuses on the interference effect of prior learning, might contribute to assisting students with fraction misconceptions. It was suggested that the effectiveness of the intervention be examined in comparison to a traditional intervention program. Given the effect of performance success on self-efficacy it was also suggested that the impact the intervention had on mathematical self-efficacy should be investigated.

The first literature review chapter, Chapter 2 (Literature Review: Fractions and Self-Efficacy), considered some of the complexities of fraction understanding, and pointed out the fact that despite the many studies highlighting the common misconceptions, students are still encountering difficulties when working with fractions in the classroom. Despite establishing the necessity to master fractions, students are still forming misconceptions leading to further problems if they carry these misconceptions through to the high school years. The difference between careless mathematical errors and those that are repeated erroneous misconceptions was examined. The concept of a fraction was examined closely, and it was established which fraction topics are taught, the development of fraction knowledge, and where fractions are positioned within the Australian Curriculum. The second purpose of Chapter 2 was to examine the role self-efficacy played in the learning process, and the links between self-efficacy, motivation, and engagement were discussed.

In Chapter 3 (which addressed Proactive Inhibition and the Old way / New Way strategy) it was discussed how errors can arise in human learning endeavours. After a review of the literature, it was suggested that errors are the product of previous experience; are causally determined, and often systematic; and are persistent until there is an intervention. The purpose of Chapter 3 was to further develop the knowledge of errors and error patterns in fraction computation and to consider the potential for the Old Way/New Way strategy to be an effective intervention program for the remedial learning of fractions.

In Chapter 4 (Research Design), the design utilised to achieve the aims and objectives of the research was outlined and discussed. To further develop the knowledge of errors and error patterns in fraction computation, and to determine the relative effectiveness of the Old Way/New Way strategy compared to a traditional intervention program for the remedial learning of fractions, a pilot study was conducted to trial the instruments. This led to a refinement of instruments, and the final instruments used in the main study were presented. The key instruments for the study consisted of the Fractions Diagnostic Test, and the Self-Efficacy Questionnaire, and these were given to all students in Year 7, 8, and 9. Based on the results of The Fractions Diagnostic Test, students with fraction misconceptions were invited to participate in one of two fractions remediation programs. The "error" participants also participated in a fractions diagnostic post-test, and a further self-efficacy test following intervention. A delayed retention test was used in the year following the intervention programs to determine the lasting effects of remediation.

In Chapter 5 (Results: Fraction Understanding) the conceptual understanding of fractions of the whole cohort was examined through questions related to the part-whole, operator, quotient, and measure sub-constructs. The students in this study exhibited many common misconceptions. Procedural understanding was determined with respect to the four operations of fractions: addition, subtraction, multiplication, and division, and the students' procedural competence was further examined through questions about equivalence and simplification. Many of the misconceptions demonstrated in the student samples of the procedural questions in this study were common errors reported by previous research; however, some errors in procedure were unique to this study. Common errors exhibited by students in this study included adding the numerator and adding the denominator in fraction addition (and similarly for subtraction), regardless of whether the denominators were like or unlike. Some students in this study also completed correct addition and subtraction when the denominators were the same, but then added both numerators and denominators when the denominators were different. Unique errors in this study were mostly found in the processes of fraction division and fraction multiplication. One student in this study inverted the second fraction to multiply fractions, and another student also inverted the fraction when multiplying by a whole number.

Chapter 6 (Results: Intervention Programs) looked at the effect of the two intervention programs (Traditional and O/N) on student understanding of the operations. Chapter 6 detailed and discussed the student fractions test data analysis from each stage of the research. The Fractions Diagnostic Test was used to ascertain which students had misconceptions of fraction computation, as determined by repeated errors. Comparisons between the participants in the "no error" and the "error" groups (O/N and Traditional Intervention) were presented as well as comparisons in effectiveness of the intervention programs for the experimental groups. Results in Chapter 6 highlighted the relative effectiveness of intervention, with the Delayed Retention Test results of the Experimental Group total score comparable to the results of the Control Group total score, indicating that the intervention had enabled the intervention students to "catch up" to their peers. Despite the lack of a distinction between the effectiveness of the two intervention programs, students in the O/N group had a shift in focus of their journal entries, from very detailed descriptions of what they learnt each lesson to only solving the problems and very few entries towards the end of the intervention trials. This suggests their understanding improved throughout the intervention and they spent more time focusing on the algorithms and less time writing about what their learning.

In Chapter 7 (Results: Self-Efficacy) the results of the self-efficacy questionnaire data analysis from the pre-test and post-test stages of the research were presented and discussed. The questionnaire was designed to measure the three domains of functioning: the affective, the cognitive, and the conative. Scores for each of the domains, as well as total self-efficacy scores from pre- and post-testing were discussed. General findings of this research suggested that the students believed that success is attributed to hard work and anyone can improve. The conative domain scores from the participants in the pre-test were the highest, and those from the cognitive domain were the lowest. Students in the "no error" group had a significantly higher mean self-efficacy score than those "error" group participants, reflecting other studies that have showed that students who are more successful generally have higher self-efficacy. Comparisons of self-efficacy score was made between the two intervention groups. The traditional intervention group had a significantly higher affective mean score at the pre-test stage. There was no significant difference in score at the post-test stage, suggesting that the O/N had a greater improvement in the affective domain after intervention. The Traditional Intervention Group had a slightly higher conative score than the O/N group at the pre-test stage. Both groups reported significant increases in conative domain mean scores at the post-test stage, suggesting an improvement in their directed effort. As there was no significant difference between the two groups after intervention, this highlights a greater improvement by the O/N group. Intervention thus had a positive effect on both groups' conation, suggesting that intervention enabled the students to improve their disposition to strive to learn, and gave them more inclination to plan, monitor, and evaluate their work. Students in the O/N group reported greater confidence in solving fractions compared to the students in the Traditional group who focused their journal entries on describing a better understanding of fractions and mathematics in general. These journal entries provide an insight into the possible reasons for change for the two intervention groups. The O/N group reported an increase in having the confidence to try an algorithm they may not have previously attempted and they reported a greater confidence in their ability to solve the problems. Qualitative responses from the students in the Traditional group suggest improvement was due to a better understanding of fractions in general, greater enjoyment when completing fraction algorithms and a better understanding of fractions. Journal entries were only sought from students in the intervention groups. There were no comparisons to students in the control group, who did not participate in intervention.

The results of this study highlight the prevalence of errors and misconceptions among high school students with fraction computations. Based on the results, we know that student fraction misconceptions are common, but that intervention does make a difference. The main aim of the study was to determine whether the Old Way / New Way remediation strategy was more effective than traditional intervention in "treating" fraction misconceptions. The O/N group had a significant improvement in their procedural type questions (Question 8 on the test), but did not display a higher total score than the traditional intervention group at the post-test stage. The improvement of the O/N group on procedural questions was not surprising, as it was expected that the O/N intervention strategy would be most effective for the procedural type questions the students encountered in

Question 8 of the Fractions Diagnostic Test. The results showed that intervention makes a difference but that O/N was not significantly better than traditional remediation across all aspects of fractions. Both intervention groups reported similar delayed retention test scores to the "no error" group, which indicates that they were able to improve their understanding after intervention and were no different to the "no error" group the following year.

It has been established that self-efficacy does improve after intervention but that improvement occurs at the sub-domain level. Context-specific, and even task-specific, self-efficacy beliefs were found to have greater predictive power for future achievement. Students in this study reported greater self-efficacy on fractions specific items on the questionnaire, after intervention. After intervention, it was found that the perceived mastery experience was a powerful source of students' mathematics self-efficacy. Students who felt they had mastered skills and succeeded at the Fractions Diagnostic Test experienced an improvement in their self-efficacy.

8.3 What does this mean?

Despite significant research into teaching fractions for understanding over the past few decades, students are still exhibiting misconceptions and errors when working with fractions, and they report that they do not understand fractions, nor do they enjoy learning about fractions. The Fractions Diagnostic Test in this study elicited many of the common misconceptions found in past research but there were errors and misconceptions unique to this study. The majority of the fraction addition and subtraction misconceptions in this study are common to many other studies.

For example, many students still add/subtract the numerator and add/subtract denominator, for both like and unlike denominators, as previously established. The mixed fraction multiplication in this study proved to be the most difficult for students, with the majority of students treating the whole numbers separately when calculating the answer. Despite some similarities with past research findings, the main differences in the results of this study were in the responses from the multiplication and division questions. For example, in the mixed fraction multiplication problem some students knew they needed to convert the fraction to an improper fraction but instead of converting and then carrying out the multiplication, they used the conversion of the first fraction as the numerator of the final answer and the conversion of the second fraction as the denominator of the answer ($1\frac{3}{4} \times 2\frac{5}{6} = \frac{7}{17}$). In the division questions, a found misconception was where a student converted the fractions to common denominators but then subtracted the numerators ($\frac{3}{4} \div \frac{1}{8} = \frac{6}{8} - \frac{1}{8} = \frac{5}{8}$ and $\frac{1}{6} \div \frac{2}{3} = \frac{1}{6} - \frac{4}{6} = \frac{-3}{6}$).

There is still much to learn about students' understanding of fractions and the procedures they adopt when faced with a variety of fraction algorithms. Despite reiterating the knowledge of students' understanding from past research, this study has examined fraction understanding in more detail. Coupled with the comparison of two intervention programs, and the effect the interventions had on fraction understanding, and the knowledge that a student's self-efficacy can be influenced by task-specific exercises, we can now narrow the focus of remediation and strengthen the impact with a multifaceted approach. Fractions are key in many areas of mathematics and numeracy, assisting students to gain confidence and improved procedural ability is important. The Fractions Diagnostic Test can potentially provide useful baseline data for teachers about students' fraction competency. Further use and investigation of the O/N strategy may provide additional methods for teachers to assist students in improving skills – in a way that traditional methods have failed to do so.

By using the newly developed Fractions Diagnostic Test from this study, a targeted intervention could be tailored to the individual student, whilst monitoring the task-specific focus on self-efficacy. A student's success in fraction work should be determined not only by improvement in a test score, but by their motivation to succeed, and a directed effort and desire to strive. As established in this study, students with low achievement scores also report lower engagement and motivation for mathematics. If we could add all the pieces of information derived from this research together, it should be possible to design an effective intervention program for fraction misconceptions that also increased self-efficacy in, and therefore motivation for, mathematics.

This study highlighted how the O/N strategy is a marginally more effective technique for mathematical procedural/process items, rather than conceptual "retraining" approach. This is both a positive and a limitation to the study as the strategy is a time-efficient and effective approach, however, the strategy is limited to changing only certain types of errors. The strategy is effective in overcoming the effects of proactive inhibition and allows the individual to remember mathematical processes, yet the strategy will not aid in developing the conceptual understanding of subject matter.

8.4 Limitations

A major strength of the study was the multifaceted approach to fraction understanding among secondary students. This study was able to elicit extensive information about students' approach to fraction computations. Rigorous comparisons of the effect of intervention on fraction understanding were also made across three year levels of secondary students, and important information relating to students' mathematics self-efficacy was gathered. The researcher-designed Fractions Diagnostic Test and fraction-specific self-efficacy questionnaire will be effective instruments to use to examine students' understanding of fractions based on the sub-constructs of rational number, and are aligned to the Australian Curriculum. The instruments are appropriate for use across the secondary year levels of 7, 8, and 9 and could be used for students participating in a modified curriculum in Year 10.

Despite the strengths of this study and the positive outcomes as a result, as with other studies, this research has its limitations in terms of sampling, instruction, data collection, and analysis. Some of these limitations were known upfront and have been acknowledged in the Methodology chapter, but further limitations were also discovered into the data collection phases of the research.

8.4.1 Sampling

The total number of participants in the pre-test was 361, with 83 individuals identified as having errors, and who were invited to participate in the intervention program. Forty students gave consent for participation in the intervention, with 35 completing all aspects of the study. Access to secondary school students for research of any kind is difficult and getting them to consent to extra sessions of mathematics in their spare time is even more challenging. Although it would have been ideal to have all 83 students who were identified as have fraction misconceptions participating in the study, it must be taken into account that the students signed up for remedial mathematics sessions during their lunch break. Due to the time constraints, the students were divided into the two intervention groups and the remediation was delivered concurrently as whole-group sessions. It would be interesting to note any differences to the results if the students had been dealt with on a one-to-one basis for remediation. It would be fair to assume that some of the 35 participants may not have been fully engaged in the activities due to these being conducted in their lunch break, the only break before classes resumed for the afternoon. Students who have positive and relatively high self-efficacy beliefs are more likely to be engaged in the classroom in terms of their behaviour, cognition, and motivation. Students participating in the intervention programs of this study generally had low mathematics self-efficacy and therefore may have been more disinclined to engage during the activities.

Although the information gathered from the 35 intervention participants was invaluable to the study and the results gained from their participation enabled me to contribute significantly to research in the areas of fraction understanding and self-efficacy, caution should be exercised in generalising the results. The study provided strong evidence that the entire cohort of students (n=361) performed poorly on the Fractions Diagnostic Test, despite comprehensive coverage of the rational number requirements of the

Australian Curriculum. Very few students demonstrated good computational skills related to fractions and those with low achievement scores on the test had lower mathematics self-efficacy compared to their peers. The results of the study suggest teachers should use diagnostic tools often as they allow teachers to see misconceptions more clearly, despite a student achieving sound results. A student may score well on a summative assessment but an underlying misconception may go unnoticed without purposeful diagnostic testing.

8.4.2 Instrumentation

The Fractions Diagnostic Test used in this study was researcher developed, based on suggested research relating to a balance of conceptual and procedural questions and was based on and developed in respect to the rational number sub-constructs. Specific questions were also included in response to research that determined that students perform better with familiar fractions (Gabriel et al., 2013), therefore "unfamiliar" fractions (e.g. $\frac{4}{7}$) were used in the test alongside familiar fractions (e.g. $\frac{3}{4}$) to gain a better picture of conceptual knowledge. The test was trialled in the Pilot Study and improved upon for the main study. The test was modified again for the delayed retention test the year following the intervention programs. Validity was maintained through checking links between fraction knowledge categories and the theoretical model proposed by Behr et al. (1983). Reliability for the Fractions Diagnostic Test was not ascertained in this study.

The self-efficacy questionnaire was a modified instrument, with some changes made to the wording of generic "mathematics" to be fraction-specific.

The questionnaire had a high level of internal consistency, as determined by Cronbach's alpha at the pre-test stage. The subscales also had adequate reliability.

8.4.3 Teaching Method

The Old Way /New Way strategy is regarded as a *convergent* remediation approach, focusing on computational knowledge, as the O/N approach converges toward the teaching of a specified matter. The convergent approach is highly structured, and teacher-centred; students are passive recipients of knowledge transmitted to them. Lyndon (1989) suggested remediation of erroneous errors needed to embrace the influence of proactive inhibition (PI) to be effective. Based on the interference effect of PI, the Old Way/New Way strategy had previously been demonstrated in a wide variety of applications where changes in habit, skills, and concepts are required. In mathematics interventions teachers are often looking for time-efficient methods of reteaching concepts that are not understood. The potential superiority of the O/N method lies in the short amount of time and effort required for implementation and its power to motivate students. It should be noted that the O/N approach used in this study focused on procedural methods not on conceptual understanding. The main purpose of the approach in this study was to use it as an intervention method for those students who had been taught how to do a fraction computation but who were still displaying erroneous errors in their work. The purpose of the strategy is to overcome the effects of PI via a quick intervention; it is not regarded as a teaching method. The O/N strategy should be used as an efficient *intervention* after an appropriate teaching method of the curriculum has been delivered.

The purpose of this study was to determine the effectiveness of the time efficient O/N strategy in comparison to the more time-consuming traditional remediation program for fraction understanding. These interventions were used after the delivery of a secondary school mathematics program involving fractions in the Australian Curriculum. Apart from an inclusion of, and a discussion of, an outline of the fractions curriculum content and summative assessments, this study did not examine the effectiveness of the initial teaching programs. There was no discussion of what good teaching looks like; this study was based entirely on the assumption that teachers were teaching effective programs, which achieved the desired outcomes suggested in the Australian Curriculum. Research as part of this study suggested instruction of rational number based on the sub-constructs of rational number. Comparisons were drawn from formative and summative assessments at each year level, and from curriculum documentation. There was no further scrutiny of teaching methods in the mathematics program across the year levels.

8.5 Outcomes and directions for future research

The focus of this study was on the mathematical errors and problems secondary students in an Australian secondary school had with fraction understanding. It examined the mathematics self-efficacy of the students pre- and post-intervention and examined the effectiveness of the O/N remediation strategy compared to traditional remediation. The study has:

- Provided a Fractions Diagnostic Test, that allows teachers to gain a formative assessment of students' fraction understanding, with questions that can highlight the common errors and misconceptions;
- Provided further understanding of how students make sense of fractions, highlighting the common errors and misconceptions;
- Informed the design of intervention programs related to fraction misconceptions; and
- Provided a preliminary assessment instrument for mathematics self-efficacy for students working with fractions, that could be transferable to other curriculum areas.

This study revealed that difficulties with fractions continue to plague secondary school students and, despite a considerable amount of research in this area, fraction misconceptions are common. Students continue to make errors when working with fractions in the secondary school years, regardless of explicit instruction in fraction computation aligned to the Australian Curriculum. Conceptual understanding and procedural fluency were positively correlated but the influence one had on the other was unable to be established. Some students who made errors on the Fractions Diagnostic Test displayed some conceptual understanding, determined by the method they used for solving problems of fraction operations and the explanations they provided with their calculations. Other students who made errors used incorrect processes to solve fraction operation questions, however, their level of conceptual fraction understanding was unable to be established. Behr, Lesh, Post, and Silver (1983) concluded that mastering the interpretations of fractions contributes towards acquiring proficiency in the operations of fractions, and that students' performance on the operations of fractions required both procedural fluency and conceptual understanding of the operations. The Behr, Lesh, Post, and Silver model illustrated the importance of the part-whole/partitioning sub-construct, which they consider to be fundamental for developing understanding of the other four sub-constructs of *ratio*, *operator*, *quotient*, and *measure*.

Further research conducted as a result of this study should be based around an examination of the achievement standards of the Australian Curriculum for fractions and rational number at each year level. I would argue that effective instruction that develops conceptual knowledge of fractions is crucial to students' understanding and achievement outcomes of fractions. Secondary students are struggling with fraction concepts, yet they are expected to understand all procedures for fraction operations by the end of Year 7. Many students at this stage do not understand the concepts underlying the operations and are therefore making mistakes without knowing why. It is recommended that the interplay between conceptual and procedural understanding of fractions be examined more closely. Conceptual understanding of fractions is crucial to procedural fluency, and although this study found a strong correlation between students' conceptual and procedural knowledge, it did not specifically examine the significance of the relationship between the two. Further examination of whether conceptual understanding can develop independently of procedural knowledge, and vice versa, is recommended.

In Australia, we need to look more closely at the teaching of fractions in the primary years so we can prepare students more effectively for the desired achievement outcomes. A research agenda might include an examination of the teaching of fraction concepts as part of the Australian Curriculum. An examination of how the development of scope and sequence curriculum documents fit with the rational number sub-constructs, and a closer examination of conceptual and procedural teaching for fraction understanding across all year levels is recommended. This study was able to report on students' performance on questions related to each of these sub-constructs, but closer examination of the interaction between these items is recommended. Diagnostic testing can determine a student's understanding at each level of the fraction construct and this, in turn, can provide information about the appropriate intervention method to use.

Closer examination of the use of number lines and some consideration for the use of empty number lines would be an advantage for the teaching of fractions. The empty number line is simply a line without regular intervals on which students record their thoughts for problem-solving. The power of the empty number line is that it builds number sense in the students, causing the focus to shift from memorising to the students making up their own series of increments on the number line.

Based on the findings of this study, it is recommended that further research be undertaken to examine the relative effectiveness of one-to-one intervention. Is one-to-one intervention more effective than whole-group intervention? Is there a difference between one-to-one intervention using traditional remediation compared to the O/N strategy?

8.6 Recommendations

Findings from this study have implications for: the teaching of fractions, the diagnosis of fraction misconceptions, the type of intervention to use with repeat fraction errors, and for students' mathematical self-efficacy. This study has expanded the research area of fraction understanding to support the knowledge of what students do, and how to remedy habitual errors in a practical way for increased understanding and increased self-efficacy. The following recommendations are made, based on the findings of this study:

- Examination of fractions curriculum in regard to the construct of rational number (school and/or Australian Curriculum)
- Recommendation that a good conceptual base of fractions is achieved before explicit procedural computations
- Diagnosis and intervention occur regularly to reduce fraction misconceptions

Future research should include:

- An examination of teaching methods of fractions, particularly at the primary level; and
- A closer examination of which conditions are conducive to the use of the O/N strategy, including the effectiveness of one-on-one intervention compared to whole group.

This thesis has focused on students' fraction understanding and the effects of remedial instruction on understanding, computational skills, and self-efficacy. One aim of the study was to elicit, and make visible, fraction computation misconceptions of secondary students, supporting the literature that achieving a depth of understanding in fractions is both complex and difficult. It was found that students do not construct meaning in isolation; rather, they try to make sense of new ideas based on what they already know. It was found that students often do not remember which procedural processes to use when doing fraction computation and this, coupled with a lack of deep understanding, means they often do not experience success with fractions. As a result, students become despondent about their ability and achievement in the topic, leading to low self-efficacy. The outcomes of the research have been used as a basis for the formulation of a number of conclusions and recommendations for future research. Such a research agenda would further advance our capacity to better help students learn fraction concepts and operations successfully.

References

Abdullah, A., Abidin, N. & Ali, M. (2015). Analysis of students' errors in solving higher order thinking skills (HOTS) problems for the topic of fraction. *Asian Social Science,* 11(21), 133-142.

Anderson, M. C., & Neely, J. H. (1996). Interference and inhibition in memory retrieval. In E. L. Bjork & R. A. Bjork (Eds.), *Handbook of perception and cognition* (2nd ed., pp. 237-313). San Diego, CA: Academic Press.

Ashlock, R. B. (1990). *Error patterns in computation: A semi-programmed approach* (5th ed.). New York: Macmillan.

Ashlock, R. B. (1994). *Error patterns in computation* (6th ed.). Englewood Cliffs, NJ: Prentice Hall.

Ashlock, R. B. (2010). *Error patterns in computation: Using error patterns to help each student learn* (10th ed.). Boston: Allyn & Bacon.

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2015). *About ICSEA.* Retrieved from http://www.myschool.edu.au

Australian Curriculum, Assessment and Reporting Authority (ACARA). (2017). *Australian Curriculum: Mathematics.* Retrieved from http://www.australiancurriculum.edu.au/Mathematics

Ausubel, D. P. (1968). *Educational psychology: A cognitive view* (2nd ed.). New York: Holt, Rinehart & Winston.

Bailey, D. H., Zhou, X., Zhang, Y., Cui, J., Fuchs, L. S., Jordan, N. C., Gersten, R., & Siegler, R. S. (2012). Development of fraction concepts and procedures in U.S. and Chinese children. *Journal of Experimental Child Psychology*, *129*, 68-83.

Baker, S., Gersten, R., & Lee, D-S. (2002). A synthesis of empirical research on teaching mathematics to low-achieving students. *The Elementary School Journal,* 103(1), 51-73.

Ball, D. L. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *The Elementary School Journal,* 93(4), 373-397.

Bandura, A. (1977). Self-efficacy: Towards a unifying theory of behavioural change. *Psychology Review, 84*, 191-215.

Bandura, A. (1994). Self-efficacy. In V. S. Ramachaudran (Ed.), *Encyclopedia of human behavior* (Vol. 4, pp. 71-81). New York: Academic Press. (Reprinted in H. Friedman [Ed.], *Encyclopedia of mental health*. San Diego: Academic Press. 1998).

Bandura, A. (1986). *Social foundations of thought and action: A social cognitive theory*. Englewood Cliffs, NJ: Prentice-Hall.

Bandura, A. (1995). *Self-efficacy in changing societies*. New York: Cambridge University Press.

Bandura, A. (1997). *Self-efficacy: The exercise of self-control.* New York: W. H. Freeman.

Bandura, A. (1978). The self-system in reciprocal determinism. *American Psychologist*, *33*, 344-358.

Bargh, J. A., & Chartrand, T. L. (1999). The unbearable automaticity of being. *American Psychologist*, *54*(7), 462-479.

Baroody, A, J., & Hume, J. (1991). Meaningful mathematics instruction: The case of fractions. *Remedial and Special Education, 12*(3), 54-68.

Baturo, A. R., & Cooper, T.J. (1995). Strategies for comparing decimal numbers with the same whole-number part. In B. Atweh & S. Flavel (Eds.) *Proceedings of the eighteenth annual conference of the Mathematics Education Research Group of Australasia Incorporated*, pp. 73-79). Darwin, NT: MERGA.

Baxter, P., & Dole, S. (1990). Working with the brain, not against it: Correction of systematic errors in subtraction. *British Journal of Special Education, 17*, 1, 19-22.

Baxter, P., Lyndon, E. H., & Dole, S. (1999). *Old habits no longer die hard: Accelerating the development of skilled performance using a novel approach to error and technique correction and habit unlearning.* Paper presented at the 34th Annual Conference of the Australian Psychological Society, Hobart, Tasmania.

Baxter, P. (2000). The way ahead: Old Way/New Way and mediational learning. *Classroom*, *7*, 12-13.

Baxter, P., Lyndon, H., Dole, S., & Battistutta, D. (2004). Less pain, more gain: Rapid skill development using old way new way. *Journal of Vocational Education and Training,* 56(1), 21-50.

Behr, M. J., Harel, G., Post, T. R., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296-333). NY: Macmillan Publishing.

Behr, M. J., Khoury, H., Harel, G., Post, T. R., & Lesh, R. (1997). Conceptual units analysis of pre-service elementary school teachers' strategies on a rational-number-as-operator task. *Journal for Research in Mathematics Education*, *28*, 48-69.

Behr, M. J., Lesh, R., Post, T. R., & Silver, E. A. (1983). Rational numbers concepts. In R. Lesh, & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 91-125). New York: Academic Press.

Behr, M. J., Wachsmuth, I., Post, T. R., & Lesh, R. (1984). Order and equivalence of rational numbers: A clinical teaching experiment. *Journal for Research in Mathematics Education*, *15*, 323-341.

Bell, A. W. (1984). Short and long term learning – experiments in diagnostic teaching design. In *Proceedings of the Eighth International Conference for the Psychology for Mathematics Education*, Sydney.

Bezuk, N., & Bieck, M. (1993). Current research on rational numbers and common fractions: Summary and implications for teachers. In D. Owens (Ed.), *Research ideas for the classroom: Middle grade mathematics.* NY: MacMillan.

Bezuk, N., & Cramer, K. (1989). Teaching about fractions: What, when and how? In P. Trafton (Ed.), *National Council of Teachers of Mathematics 1989 Yearbook: New directions for elementary school mathematics* (pp.156-167). Reston, VA: National Council of Teachers of Mathematics.

Billett, S. (2012). Errors and learning from errors at work. In J. Bauer & C. Harteis (Eds.), *Human fallibility: The ambiguity of errors for work and learning* (pp. 17-32). Netherlands: Springer.

Boaler, J. (2016). *Mathematical mindsets: Unleashing students' potential through creative math, inspiring messages and innovative teaching*. San Francisco, CA: Jossey-Bass.

Bonne, L. (2012). *The effects of primary students' mathematics self-efficacy and beliefs about intelligence on their mathematics achievement: A mixed-methods intervention study* (Unpublished PhD thesis). Victoria University of Wellington, Wellington, New Zealand.

Borasi, R. (1987). Exploring mathematics through the analysis of errors. *For the Learning of Mathematics*, *7*(3), 2-8.

Borasi, R. (1994). Capitalising on Errors as "Springboards for Inquiry": A Teaching Experiment. *Journal for Research in Mathematics Education, 25,* 2, 166-208.

Bouchard, T. J. (1976). Unobtrusive measures: An inventory of uses. *Sociological Methods and Research, 4*(3), 267-300.

Brousseau, G., Brousseau N., & Warfield, V. (2004). Rationals and decimals as required in the school curriculum part 1: Rationals as measurement. *The Journal of Mathematical Behavior, 23*(1), 1-20.

Brown, J. S., & Van Lehn, K. (1982). Towards a generative theory of "bugs". In T. P. Carpenter, J. M. Moser, and T. A Romberg (Eds.), *Addition and subtraction – A cognitive perspective* (pp. 117-153). Hillsdale, NJ: Lawrence Erlbaum Associates.

Brownell, W. A. (1945). When is arithmetic meaningful? *Journal of Educational Research, 38*(3), 481-498.

Byrnes, J. P., & Wasik,B. A. (1991). Role of conceptual knowledge in mathematical procedural learning. *Developmental Psychology, 27*(5), 777-786.

Carpenter, T. P., Fennema, E., & Romberg, T. A. (1993). *Rational numbers: an integration of research.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Carpenter, T. P., Fennema, E., & Franke, M. L. (1996). Cognitively Guided Instruction: A knowledge base for reform in primary mathematics instruction. *Elementary School Journal, 97*(1), 1-20.

Cates, G. L., & Rhymer, K. N. (2003). Examining the relationship between mathematics anxiety and mathematics performance: An instructional hierarchy perspective. *Journal of Behavioural Education, 12*, 23-34.

Charalambous, C. Y., & Pitta-Pantazi, D. (2005). Revisiting a theoretical model on fractions: Implications for teaching and research. In Chick, H. L., & Vincent, J. L. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp-233-240. Melbourne: PME.

Charalambous, C. Y., & Pitta-Pantazi, D. (2006). Drawing on a theoretical model to study students' understanding of fractions. *Educational Studies in Mathematics, 64*, 293-316.

Chen, P. P. (2003). Exploring the accuracy and predictability of self-efficacy beliefs of seventh-grade mathematics students. *Learning and Individual Differences, 14*(1), 77-90.

Chick, N. L., Karis, T., & Kernahan, C. (2009). Learning from their own learning: How metacognitive and meta-affective reflections enhance learning in race-related courses. *International Journal for the Scholarship of Teaching and Learning, 3*(1), 1-26.

Chinnappan, M., & Lawson, M. (2002). Year 3 children's understanding of fractions: Are we making progress? In B. Barton, K. Irwin, M. Pfannkuch & M. Thomas (Eds.), *Mathematics education in the South Pacific, (*Proceedings of the 25th Conference of the Mathematics Education Research Group of Australasia, 7-10 July, vol. 1, pp. 195-

202). Auckland, New Zealand: Mathematics Education Research Group of Australasia.

Clarke, D. M. (2008). Fractions as division: The forgotten notion? *Australian Primary Mathematics Classroom, 11*(3), 4-10.

Clarke, D. M., Roche A., & Mitchell, A. (2008). Ten practical, research-based tips for making fractions come alive (and make sense) in the middle years. *Mathematics Teaching in the Middle School, 13*(7), 373-380.

Clarke, D. M., Roche, A., & Mitchell, A. (2007). Year six fraction understanding: A part of the whole story. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential research, essential practice.* (Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia, Vol. 1, pp.207-216). Hobart: MERGA.

Cobb, P. (2007). Putting philosophy to work: Coping with multiple research perspectives. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 3-37). Charlotte, NC: Information Age Publishing.

Cole, P., & Chan, L. K. (1990) *Methods and strategies for special education*. Sydney: Prentice Hall.

Collins, K. M. T., Onwuegbuzie, A. J., & Sutton, R. B. (2012). Securing a place at the table: Introducing legitimation criteria for the conduct of mixed research. *Amercian Behavioral Scientist, 56*, 849-865.

Connell, M., & Peck, D. (1993). Report of a conceptual change intervention in elementary mathematics. *Journal of Mathematical Behaviour, 12*, 329-350.

Coward, L. A. (2014). Brain computational primitives. *Procedia Computer Science, 41*, 164-175.

Cox, L. S. (1975). Diagnosing and remediating systematic errors in addition and subtraction computations. *Arithmetic Teacher, 22*(4), 151-157.

Creswell, J. W., & Garrett, A. L. (2008). The "movement" of mixed methods research and the role of educators. *South African Journal of Education, 28,* 321-333.

Creswell, J.W., & Plano Clark, V. L. (2007). *Designing and conducting mixed methods research.* Thousand Oaks, CA: Sage.

Creswell, J. W. (2009). *Research design: Qualitative, quantitative, and mixed methods approaches.* Los Angeles: Sage.

Darby, K. P., & Sloutsky, V. M. (2015). The cost of learning: Interference effects in memory development. *Journal of Experimental Psychology, 144*(2), 410-431.

Dawson, C., & Lyndon, H. (1997). Conceptual mediation: A new perspective on conceptual exchange. *Research in Science Education, 27,* 157-173.

Dewey, J. (1931). The development of pragmatism. In H.S. Thayer (Ed.), *Pragmatism: The classic writings.* Indianapolis: Hackett.

Dole, S. (1993). Error patterns and subtraction knowledge development: A comparison of methods. In J. Novak (Ed.), *Proceedings of the Third International Seminar on Misconceptions and Educational Strategies in Science and Mathematics,* (pp. 3-24). Cornell University, New York.

Dole, S. (1999). *Percent knowledge: Effective teaching for learning, relearning and unlearning* (Unpublished doctoral dissertation). Queensland University of Technology, Australia.

Dole, S. (2003). Applying psychological theory to helping students overcome learned difficulties in mathematics: An alternative approach to intervention. *School Psychology International, 24,* 95-114.

Dole, S. (1992). New ways for old – systematic computation errors and remediation. *Teaching Mathematics, 17*(3), 18-21.

Dowker, A., Sarkar, A., & Looi, C. Y. (2016). Mathematics anxiety: What have we learned in 60 years? *Frontiers in Psychology*, *7*, 508. doi:1.3389/fpsyg.2016.00508

Durkin, K., & Rittle-Johnson, B. (2014). Diagnosing misconceptions: Revealing changing decimal fraction knowledge. *Learning and Instruction, 37, 2*1-29. doi: 10.1016/j.learninstruc.2014.08.003

Dweck, C. S. (2006). *Mindset: The new psychology of success.* New York: Random House.

Eason, R. L., Smith, T. L., & Plaisance, E. (1989). Effects of proactive interference on learning the tennis backhand stroke. *Perceptual and Motor Skills, 68*(3,), 923-930.

Eccles, J. S., Wigfield, A., & Schiefele, U. (1998). Motivation to succeed. In W. Damon & N. Eisenberg (Eds.), *Handbook of child psychology: Social, emotional, and personality development* (pp. 1017-1095). Hoboken, NJ: John Wiley & Sons Inc.

Eichelmann, A., Narciss, S., Schnaubert, L., & Melis, E. (2012). Typical errors in addition and subtraction of fractions: A review of empirical error analysis. *Journal of Mathematics Didactics, 33*(1), 29-57.

Empson, S. B., & Levi, L. (2011). *Extending children's mathematics: Fractions and decimals: Innovations in cognitively guided instruction.* Portsmouth, NH: Heinemann.

English, L., & Sriraman, B. (2005). Theories of mathematics education. In H. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th Annual meeting of the*

International Group of Psychology of Mathematics Education (Vol. 1, pp. 170-172). Melbourne, Australia.

Fazio, L., & Siegler, R. (2010). *Teaching fractions*. Rotterdam: International Academy of Education.

Flavell, J. H. (1987). Speculation about the nature and development of metacognition. In F. E. Wernert & R. H. Kluwe (Eds.), *Metacognition, motivation and understanding* (pp. 21-29). Hillsdale, NJ: Lawrence Erlbaum Associates.

Fisher, B. J., Bruce, M. E., & Greive, C. (2007). The entry knowledge of Australian pre-service teachers in the area of phonological awareness and phonics. In A. Simpson (Ed.), *Future directions in literacy: International conversations 2007*. University of Sydney. Retrieved from: http://ses.library.usyd.edu.au/bitstream/2123/2330/1/FutureDirections_Ch5.pdf

Fitts, P. M., & Posner, M. I. (1967). *Human performance*. Belmont, CA: Brooks/Cole Publishing Company.

Gable, R.A., Enright, B. E., & Hendrickson, J. M. (1991). A practical model for curriculum-based assessment and instruction in arithmetic. *Teaching Exceptional Children, 24*, 6-9.

Gabriel, F., Coche, F., Szucs, D., Carette, V., Rey, B., & Content, A. (2013). A componential view of children's difficulties in learning fractions. *Frontiers in Psychology, 4*, 1-12.

Getenet, S., & Callingham, R. (2017). Teaching fractions for understanding: Addressing interrelated concepts. In A. Downton, S. Livy, & J. Hall (Eds.), *40 years on: We are still learning! Proceedings of the 40th Annual*

Conference of the Mathematics Education Research Group of Australasia (pp. 277-284). Melbourne: MERGA.

Giannouli, V. (2013). Number perseveration in healthy subjects: Does prolonged stimulus exposure influence performance on a serial addition task? *Advances in Cognitive Psychology, 9*(1), 15-19.

Gomez, D., Jimenez, A., Bobadilla, R., Reyes, C., & Dartnell, P. (2014). Exploring fraction comparison in school children. In S. Oesterle, P. Liljedahl, C. Nicol, & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA,* Vol. 3, pp. 185-192. Vancouver, Canada: PME.

Gould, P. (2005). Really broken numbers. *Australian Primary Mathematics Classroom, 10*(3), 4-10.

Graham, S., & Weiner, B. (1996). Theories and principles of motivation. In D. C. Berliner & R. C. Calfee (Eds.), *Handbook of educational psychology* (pp. 63-84). New York: Simon & Schuster.

Greene, J. C., Caracelli, V. J., & Graham, W. F. (1989). Toward a conceptual framework for mixed-methods evaluation design. *Educational Evaluation and Policy Analysis, 11*(3), 255-274.

Hall, M., & Ponton, M. (2002). *A comparative analysis of mathematics self-efficacy of developmental and non-developmental freshman mathematics students.* Paper presented at the meeting of the Louisiana/Mississippi Section of the Mathematics Association of America, Natchitoches, LA. doi:10.1.1.465.635

Hallett, D., Nunes, T., & Bryant, P. (2010). Prospective teachers' conceptual and procedural knowledge in mathematics: The case of algebra. *American Journal of Educational Psychology, 102*, 395-406.

Hampton, N. Z., & Mason, E. (2003). Learning disabilities, gender, sources of efficacy, self-efficacy beliefs, and academic achievement in high school students. *Journal of School Psychology, 41*(2), 101-112.

Hanin, Y., Korjus, T., Jouste, P., & Baxter, P. (2002). Rapid technique correction using Old Way New Way: Two case studies with Olympic athletes. *Sport Psychologist, 16*, 79-99.

Hanin, Y., Malveda, M., & Hanina, M. (2004). Rapid correction of start technique in an Olympic-level swimmer: A case study using Old Way/New Way. *Journal of Swimming Research*, *16*, 11-17.

Hart, K. (1981). *Children's understanding of mathematics:* 11-16. London: Murray.

Hatano, G. & Inagaki, K. (1986). Two courses of expertise. In H. W. Stevenson, H. Azuma, & K. Hakuta (Eds.), *A series of books in psychology. Child development and education in Japan* (pp. 262-272). New York: Times Books.

Hawker, S., & Cowley, C. (1998). *The Oxford quick reference dictionary and thesaurus.* Oxford: Oxford University Press.

Heiberger, R., & Robbins, N. B. (2014). Design of diverging stacked bar charts for likert scales and other applications. *Journal of Statistical Software, 57*(5), 1-32.

Henderson, R., Higgs, G., Lyndon, E. H., Wilkinson, D., & Yates, G. (1999). *Conceptual mediation program in science and mathematics: effects on motivational indices and strategy awareness.* Paper presented at the Department of Education, Training and Employment Research Expo, Adelaide, South Australia.

Hiebert, J., & Lefevre, P. (1986). *Conceptual and procedural knowledge in mathematics: An introductory analysis.* Hillsdale, NJ: Erlbaum.

Hiebert, J., Wearne, D., & Taber, S. (1991). Fourth graders' gradual constructions of decimal fractions during instruction using different physical representations. *Elementary School Journal, 91*(4), 321-341.

Hollnagel, E. (1993). The phenotype of erroneous actions. *International Journal of Man-Machine Studies, 39,* 1-32.

Huinker, D. (2002). Examining dimensions of fraction operation sense. In B. Litwiller (Ed.), *Making sense of fractions, ratios, and proportions.* Reston, VA: National Council of Teachers of Mathematics.

Huitt, W., & Cain, S. (2005). An overview of the conative domain. In *Educational Psychology Interactive,* pp. 1-20. Voldosta, GA: Voldosta State University.

Johnson, R. B., Onwuegbuzie, A. J., & Turner, L. A. (2007). Toward a definition of mixed methods research. *Journal of Mixed Methods Research, 1,* 112-133.

Jonides, J., & Nee, D. E. (2006). Brain mechanisms of proactive interference in working memory. *Neuroscience, 139,* 181-193.

Kephart, N., C. (1960). *The slow learner in the classroom.* Columbus, OH: Charles E. Merrill.

Keppel, G., & Underwood, B. J. (1962). Proactive inhibition in short-term retention of single items. *Journal of Verbal Learning and Verbal Behaviour, 1*(3), 153-161.

Keppel, G., & Zedeck, S. (1989). *Data analysis for research designs: Analysis of variance and multiple regression/correlation approaches.* New York: Times Books.

Kerslake, D. (1986). *Fractions: Children's strategies and errors: A report of the strategies and errors in secondary mathematics project*. Windsor, UK: NFER-Nelson.

Kahn, N. (2018). *The impact of mindset and grit on academic success in math and science*. (Unpublished doctoral dissertation). New York: Hofstra University.

Kieren, T. (1976). Personal knowledge of rational numbers: Its intuitive and formal development. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 53-92). Reston, VA: National Council of Teachers of Mathematics.

Kieren, T. E. (1981). *Five faces of mathematical knowledge building.* Edmonton: University of Alberta.

Kieren, T. E. (1995). Creating spaces for learning fractions. In J. T. Sowder & B. P. Schappelle (Eds.), *Providing a foundation for teaching mathematics in the middle grades* (pp. 31-65). Albany: State University of New York Press.

Kilpatrick, J. (2014). A history of research in mathematics education. doi: 10.1007/978-94-007-4978-8_71

Kilpatrick, J., Swafford, & Findell, B. (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: The National Academies Press.

Lamon, S. (1999). *Teaching fractions and ratios for understanding: Essential content knowledge and instructional strategies for teachers*. NJ: Lawrence Erlbaum.

Lamon, S. L. (2001). Presenting and Representing: From Fractions to Rational Numbers. In A. Cuoco & F. Curcio (Eds.), *The roles of representations in school mathematics: 2001 Yearbook,* (pp. 146-165). Reston: National Council of Teachers of Mathematics.

Leinhardt, G. (1988). Getting to know: Tracing student's mathematical knowledge from intuition to competence. *Educational Pyschologist*, *23*, *2*, 119-144.

Lerman, S. (2000). A case of interpretations of social: A response to Steffe and Thompson. *Journal for Research in Mathematics Education*, *31*(2), 210-227.

Lesh, R., Post, T., & Behr, M. (1988). Proportional reasoning. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 93-118). Reston, VA: National Council of Teachers of Mathematics.

Lester, F. K. (2005). On the theoretical, conceptual, and philosophical foundations for research in mathematics education. *International Reviews on Mathematical Education*, *37*(6), 457-467.

Linnenbrink, E. A., & Pintrich, P. R. (2003). The role of self-efficacy beliefs in student engagement and learning in the classroom. *Reading and Writing Quarterly*, *19*(2), 119-137.

Longfield, J. (2009). Discrepant teaching events: Using an inquiry stance to address students' misconceptions. *International Journal of Teaching and Learning in Higher Education*, *21*(2), 266-271.

Lyndon, H. (1982). *The effects of proactive and retroactive inhibition: The Old Way/New Way methodology.* Unpublished report to the Director General of Education, South Australia.

Lyndon, H. (1989). I did it my way! An introduction to the Old Way/New Way methodology. *Australasian Journal of Special Education*, *13*(1), 32-37.

Lyndon, H. & Dawson, C. (1995). *The conceptual mediation program: a practical outcome from a novel perspective on conceptual change*. Paper presented at the 26th Annual Conference of the Australasian Science Education Research Association. Bendigo, Victoria.

Lyndon, H., & Malcolm, B. A. (1984). *The effects of proactive and retroactive inhibition: The Old Way/New Way methodology and its application to speech pathology.* Proceedings of the Annual Conference of the Australian Association of Speech and Hearing: beyond 1984, Adelaide, South Australia.

Lyndon, H. (2000). *Conceptual mediation: A new theory and a new method of conceptual change* (Unpublished doctoral dissertation). Australia: University of Adelaide.

Mack, N. K. (2001). Building on informal knowledge through instruction in a complex domain: Partitioning, units, and understanding multiplication of fractions. *Journal for Research in Mathematics Education, 32*(3), 267-295.

MacLeod, C. M. (2015). The Stroop Effect. In R. Luo (Ed.), *Encyclopedia of Colour Science and Technology*. Berlin: Springer Science.

Mamede, E., Nunes, T., & Bryant, P. (2005). The equivalence and ordering of fractions in part-whole and quotient situations. In H. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*. Melbourne, Australia: University of Melbourne.

Marcus, G. (2015, June 28). Face it, your brain is a computer. *The New York Times.* Retrieved from http://www.nytimes.com

Maschette, W. (1985). Correcting technique problems of a successful junior athlete. *Sports Coach, 9,* 14-17.

Maxwell, S. E., & Delaney, H. D. (2004). *Designing experiments and analysing data*. NJ: Lawrence Erlbaum.

May, D. K. (2009). *Mathematics self-efficacy and anxiety questionnaire.* (Unpublished doctoral dissertation. Georgia: University of Georgia). Retrieved from https://getd.libs.uga.edu/pdfs/may_diana_k_200908_phd.pdf

McTavish, T. S., & Larusson, J. A. (2014). Labeling mathematical errors to reveal cognitive states. In C. Rensing, S. de Freitas, T. Ley, & P. J. Munoz-Merino (Eds.), *Open learning and teaching in educational communities*, (pp. 446–451). Graz, Austria: Springer.

Mestre, J. (1989). *Hispanic and Anglo students' misconceptions in mathematics*. Scientific Reasoning Research Institute, University of Massachusetts. (ERIC Document Reproduction Service No. ED313192)

Miller, R. B., Greene, B. A., Montalvo, G. P., Ravindran, B., & Nichols, J. (1996). Engagement in academic work: The role of learning goals, future consequences, pleasing others, and perceived ability. *Contemporary Educational Psychology, 21, 388-422.*

Miura, I. T., Okamoto, Y., Vlahovic-Stetic, V., Kim, C. C., & Han, J. H. (1999). Language supports for children's understanding of numerical fractions: cross-national comparisons. *Journal of Experimental Child Psychology*, *74*(4), 356-365.

Morales, Z. A. (2014). *Analysis of students' misconceptions and error patterns in mathematics: The case of fractions*. Unpublished manuscript, Florida International University. Retrieved from http://digitalcommons.fiu.edu/cgi/viewcontent.cgi?article=1350&context=sferc

Moss, J., & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education, 30*(2), 122-147.

Munro, J. (2011). *The role of working memory in mathematics learning and numeracy*. Paper presented at the Memory and Learning Conference: What Works? September 1st- 2nd, Sydney: The Children's Hospital at Westmead Education and Research Institute.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics.* Reston, VA: NCTM.

National Mathematics Advisory Panel. (2009). *The final report of the national mathematics advisory panel: Foundations for success.* Jessup, MD: U.S. Department of Education.

Newstead, K., & Murray, H. (1998). Young students' constructions of fractions. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education, 3,* (pp. 295-303), Stellenbosch, South Africa: PME

Ni, Y. (2000). How valid is it to use number lines to measure children's conceptual knowledge about rational number? *Educational Psychology, 20*, 139-152.

Ni, Y., & Zhou, Y-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist*, *40,* 1, 27-52. doi: 10.1207/s15326985sep4001_3

Nicholls, G., & Ward, B. (1998). Now you see it, now you don't. *Research in Education*, *60,* 40-53.

Novillis, C. G. (1976). An analysis of the fraction concept into a hierarchy of selected subconcepts and the testing of the hierarchial dependencies. *Journal for Research in Mathematics Education, 7*, 131-144.

Nunes, T. (1992). Cognitive invariants and cultural variation in mathematical concepts. *International Journal of Behavioral Development, 15*(4), 433-453.

Obersteiner, A., Van Hoof, J., Verschaffel, L., & Van Dooren, W. (2016). Who can escape the natural number bias in rational number tasks? A study involving students and experts. *British Journal of Psychology, 107*(3), 537-555.

Olivier, A. (1989). Handling pupils' misconceptions. *Pythagoras, 21,* 9-19.

Owens, D. T., & Super, D. B. (1993). Teaching and learning decimal fractions. In D. T. Owens (Ed.), *Rsearch ideas for the classroom* (pp. 137-158). New York: Macmillan.

Padberg, F. (1986). Typical student difficulties in fraction calculations: Inventory and consequences. *Mathematics Lessons, 32*(3), 58-77.

Pajares, F. (1996a). Self-efficacy beliefs in academic settings. *Review of Educational Research, 66*(4), 543-578.

Pajares, F. (1996b). Self-efficacy beliefs and mathematical problem solving of gifted students. *Contemporary Educational Psychology, 21,* 325-244.

Pajares, F., & Graham, L. (1999). Self-efficacy, motivation constructs and mathematics performance of entering middle school students. *Contemporary Educational Psychology, 24*(2), 124-139.

Pajares, F., & Kranzler, J. (1995). Self-efficacy beliefs and general mental ability in mathematical problem-solving. *Contemporary Educational Psychology, 20*(4), 426-443.

Pajares, F., & Miller, M. D. (1994). The role of self-efficacy beliefs and self-concept beliefs in mathematical problem solving: A path analysis. *Journal of Educational Psychology, 86*(2), 193-203.

Pearn, C., & Stephens, M. (2004). Why you have to probe to discover what Year 8 students really think about fractions. In I. Putt, R. Faragher & M. McLean (Eds.), *Mathematics education for the third millenium: Towards 2010, Proceedings of the 27th annual conference of the Mathematics Education Research Group of Australasia* (pp. 430–437). Sydney, Australia: MERGA.

Pearn, C., & Stephens, M. (2007). Whole Number Knowledge and Number Lines Help to Develop Fraction Concepts. In J. Watson & K. Beswick (Eds.), *Mathematics: Essential research, essential practice.* (Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia (Vol. 2, pp. 207-216). Hobart: MERGA.

Pearn, C., & Stephens, M. (2017). Generalising fraction structures as a means for engaging in algebraic thinking. In A. Downton, S. Livy, & J. Hall (Eds.), *40 years on: We are still learning! (Proceedings of the 40th Annual Conference of the Mathematics Education Research Group of Australasia,* pp. 277-284). Melbourne: MERGA.

Peck, D. M., & Jencks, S. M. (1981). Conceptual issues in the teaching and learning of fractions. *Journal for Research in Mathematics Education, 12*(5), 339-348.

Perso, T. (1992). Making the most of errors. *Australian Mathematics Teacher, 48*(2), 12-14.

Piaget, J. (1972). Intellectual evolution from adolescence to adulthood. *Human Development, 15*(1), 1-12.

Piaget, J. (1971). The theory of stages in cognitive development. In D. R. Green, M. P. Ford, & G. B. Flamer, *Measurement and Piaget* (pp. 172-188). New York, NY: McGraw-Hill.

Pietsch, J., Walker, R., & Chapman, E. (2003). The relationship among self-concept, self-efficacy, and performance in mathematics during secondary school. *Journal of Educational Psychology, 95*(3), 589-603.

Pintrich, P. R., & De Groot, E. V. (1990). Motivation and self-regulated learning components of classroom academic performance. *Journal of Educational Psychology, 82*(1), 33-40.

Pintrich, P., & Schrauben, B. (1992). Students' motivational beliefs and their cognitive engagement in classroom academic tasks. In D. H. Schunk, & J. L Meece (Eds.), *Student perceptions in the classroom.* Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Pintrich, P., & Schunk, D. (1996). *Motivation in education: Theory, research and applications.* Englewood Cliffs, NJ: Prentice-Hall.

Post, T. R., Cramer, K., Behr, M., Lesh, R., & Harel, G. (1993). Curriculum implications of research on the learning, teaching, and assessing of rational number concepts. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 327-362). Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Pothier, Y., & Swada, D. (1983). Partitioning: The emergence of rational number ideas in young children. *Journal for Research in Mathematics Education, 14*(5), 307-317.

Prediger, S. (2006). Continuities and discontinuities for fractions: A proposal for analysing in different levels. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.), *Proceedings of the 30th Conference of the*

International Group for the Psychology of Mathematics Education, 4, 377-384. Prague: PME.

Queensland Studies Authority. (2013). *Queensland students' understanding of fractions: Evidence from the NAPLAN test results*. Retrieved from www.qcaa.qld.edu.au/publications/reports-papers/qsa.

Radatz, H. (1980). Students' errors in the mathematical learning process: A survey. *For the Learning of Mathematics, 1*(1), 16-20.

Rauff, J. V. (1994). Constructivism, factoring and beliefs. *School Science and Mathematics*, *94*(8), 421-426.

Read, G. (1987). Following the progress of children who have received "remedial" tuition. *Support for Learning, 2*(1), 52-59.

Reason, J. (1990). *Human error.* New York, NY: Cambridge University Press.

Reason, J. (2000). Human error: Models and management. *Western Journal of Medicine*, *172*(6), 393-396.

Renninger, K. A., Hidi, S., & Krapp, A. (1992). *The role of interest in learning and development.* Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Resnick, L. B. (1983). A developmental theory of number understanding. In H. P. Glingsburg (Ed.), *The development of mathematical thinking* (pp. 109-151). Orlando, FL: Academic Press.

Resnick, L. B. (1989). Developing mathematical knowledge. *American Psychologist, 44*(2), 162-169.

Resnick, L. B., Nesher, P., Leonard, F., Magone, M., Omanson, S., & Peled, I. (1989). Conceptual bases of arithmetic errors: The case of decimal fractions. *Journal for Research in Mathematics Education, 20*(1), 8-27.

Reys, R. E., Lindquist, M., Lambdin, D. V., Smith, N. L., Rogers, A., Falle, J. … Bennett, S. (2012). *Helping children learn mathematics.* Australia: Wiley.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology, 93*(2), 346-362.

Rogoff, B. (1990). *Apprenticeship in thinking: Cognitive development in social context*. Oxford: Oxford University Press.

Rorty, R. (1990). Introduction: Pragmatism as anti-representationalism. In J.P. Murphy, *Pragmatism: From Peirce to Davidson.* Boulder: Westview Press.

Rorty, R. (1991). *Objectivity, relativism and truth: Philosophical papers*. Cambridge: Cambridge University Press.

Ross, A. A., & Onwuegbuzie, A. J. (2012). Prevalence of mixed methods research in mathematics education. *Mathematics Educator, 22*(1), 84-113.

Rowell, J. A., & Dawson, C. J. (1988). What's in a solution? A look at logic and belief in problem solving. *Research in Science Education, 18*(1), 143-151.

Rowell, J. A., Dawson, C. J., & Lyndon, H. (1990). Changing misconceptions: A challenge to science educators. *International Journal of Science Education, 12,* 167-175.

Scandura, J. M. (1967). *Research in mathematics education.* Washington, DC: National Council of Teachers of Mathematics.

Schneider, M., & Siegler, R. S. (2010). Representations of the magnitude of fractions. *Journal of Experimental Psychology, 36*(5), 1227-1238.

Schoenfeld, A. H. (2000). Purposes and methods of research in mathematics education. *Notices of the American Mathematical Society, 47*(6), 641-649.

Schoenfeld, A. H. (2008). *A study of teaching: Multiple lenses, multiple views.* Reston, VA: National Council of Teachers of Mathematics.

Schunk, D. H., & Hanson, A. R. (1985). Peer models: Influence on children's self-efficacy and achievement. *Journal of Educational Psychology, 77,* 313-322.

Schunk, D. H. (1981). Modelling and attributional effects on children's achievement: A self-efficacy and analysis. *Journal of Educational Psychology, 73,* 93-105.

Schunk, D. H. (1982a). Effects of effort attributional feedback on children's perceived self-efficacy and achievement. *Journal of Educational Psychology, 74,* 548-556.

Schunk, D. H. (1982b). Progress self-monitoring: Effects on children's self-efficacy and achievement. *Journal of Experimental Education, 51*(2), 89-93.

Schunk, D. H. (1983a). Ability versus effort attributional feedback: Differential effects on self-efficacy and achievement. *Journal of Educational Psychology, 75,* 848-856.

Schunk, D. H. (1983b). Developing children's self-efficacy and skills: The roles of social comparative information and goal setting. *Contemporary Educational Psychology, 8,* 76-86.

Schunk, D. H. (1984a). Self-efficacy perspective on achievement behaviour. *Educational Psychologist, 19,* 48-58.

Schunk, D. H. (1984b). Sequential attributional feedback and children's achievement behaviours. *Journal of Educational Psychology, 76,* 1159-1169.

Schunk, D. H. (1985). Participation in goal setting: Effects on self-efficacy and skills of learning-disabled children. *Journal of Special Education, 19,* 307-317.

Shin, M., & Bryant, D. P. (2015). A synthesis of mathematical and cognitive performances of students with mathematics learning disabilities. *Journal of Learning Disabilities, 48*(1), 96-112.

Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M. … Chen, M. (2012). Early predictors of high school mathematics achievement. *Psychological Science, 23*(7), 691-697.

Siegler, R. S., & Pyke, R. A. (2013). Developmental and individual differences in understanding of fractions. *Developmental Psychology, 49*(10), 1994-2004.

Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology, 62*(4), 273-296.

Siemon, D. (2006). *Assessment for common misunderstandings*. Prepared and published electronically by the Victorian Department of Education and Early Childhood Development. Retrieved from http://www.education.vic.gov.au/studentlearning/teachingresources/maths/common/default.htm

Siemon, D., Beswick, K., Brady, K., Clarke, J., Faragher, R., & Warren, E. (2015). *Teaching mathematics: Foundations to middle years.* Australia: Oxford University Press.

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching, 77*, 20-26.

Smith, J. P. (2002). The development of students' knowledge of fractions and ratios. In B. Litwiller & G. Bright (Eds.), *Making sense of fractions, ratios, and proportions: Yearbook of the National Council of Teachers of Mathematics* (pp. 3-17). Reston, VA: National Council of Teachers of Mathematics.

Smith, E. E., Geva, A., Jonides, J., Miller, A., Reuter-Lorenz, P., & Koeppe, R. A. (2001). The neural basis of task-switching in working memory: Effects of performance and aging. *Proceedings of the National Academy of Sciences, 98*(4), 2095-2100.

Spangler (2011). *Strategies for teaching fractions: Using error analysis for intervention and assessment.* Thousand Oaks, CA: Corwin Press.

Steffe, L. P., Cobb, P., & von Glasersfeld, E. (1988). *Recent research in psychology: Construction of arithmetical meanings and strategies.* New York: Springer-Verlag Publishing.

Steinle, V. (2004). Detection and remediation of decimal misconceptions. In B. Tadich, S. Tobias, C. Brew, B. Beatty, & P. Sullivan (Eds.), *Towards Excellence in Mathematics* (pp. 460-478). Brunswick: Mathematical Association of Victoria.

Stroop, J. R. (1935). Studies of interference in serial verbal reactions. *Journal of Experimental Psychology, 18*(6), 643-662.

Tait-McCutcheon, S. L. (2008). Self-efficacy in mathematics: Affective, cognitive, and conative domains of functioning. In M. Goos, R. Brown, & K. Makar (Eds.), *Navigating currents and charting directions* (Proceedings of the 31st Annual Conference of the Mathematics Education Research Group of Australasia, Vol.2, pp. 507-513). Brisbane, Australia.

Tanner, H., & Jones, S. (2003). Self-efficacy in mathematics and students' use of self-regulated learning strategies during assessment events. In N. A. Pateman, B. J. Doherty, & J. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 275-282). Honolulu, USA: PME.

Tobias, S. (1993). *Overcoming math anxiety.* New York: W. W. Norton & Company.

Tripp, D. (1993). *Critical incidents in teaching: Developing professional judgment.* London: Routledge.

Underwood, B. J. (1966). Individual and age group predictions of item difficulty for free learning. *Journal of Experimental Psychology, 71*(5), 673-679.

Underwood, B. J. (1957). Interference and forgetting. *Psychological Review, 64*(1), 49-60.

Usher, E. L., & Pajares, F. (2009). Sources of self-efficacy in mathematics: A validation study. *Contemporary Educational Psychology, 34*(1), 89-101.

Vamvakoussi, X., & Vosniadou, S. (2010). How many decimals are there between two fractions? Aspects of secondary school students' understanding of rational numbers and their notation. *Cognition and Instruction, 28,* 2, 181-209.

Van Lehn, K. (1982). Bugs are not enough: Empirical studies of bugs, impasses and repairs in procedural skills. *The Journal of Mathematical Behavior, 3*(2), 3-71.

Vinner, S. (1997). The pseudo-conceptual and the pseudo-analytical thought processes in mathematics learning. *Educational Studies in Mathematics, 34*(2), 97-129.

Voza, L. (2011). Winning the "hundred years' war". *Teaching Children Mathematics, 18*(1), 32-37.

Waters, S., Lester, L., & Cross, D. (2014). Transition from primary to secondary school: Expectation vs experience. *Australian Journal of Education, 58*(2), 153-166.

Weaver, G., Baxter, P., & Lyndon, E. H. (2000). Changing work habits: More gain, less pain. *Australian Safety News,* October, pp. 58-59.

Wilcox, B., & Munroe, E. E. (2011). Integrating writing and mathematics. *The Reading Teacher, 64*(April), 521-529.

Williams, J. E. (1994). Gender differences in high school student's efficacy: Expectations/performance discrepancies across four subject matter domains. *Psychology in the Schools, 31*(3), 233-237.

Wittmann, E. C. (2013). The consistency of students' error patterns in solving computational problems with fractions. In B. Ubuz, C. Haser, & M. A. Mariotti (Eds.), *Proceedings of the 8th Congress of the European Society for Research in Mathematics Education* (pp. 393-402). Antalya, Turkey.

Wolters, C. A., & Rosenthal, H. (2000). The relationship between students' motivational beliefs and their use of motivational regulation strategies. *International Journal of Educational Research, 33*(7-8), 801-820.

Wong, M., & Evans, D. (2007). Improving basic multiplication fact recall for primary school students. *Mathematics Education Research Journal, 19*, 89-107.

Wright, R. J., Tabor, P. D., & Ellemor-Collins, D. (2011). *Developing number knowledge: Assessment, teaching and intervention with 7-11 year olds.* London: Sage Publications Ltd.

Yanik, H. B., Helding, B, & Flores, A. (2008). Teaching the concept of unit in measurement interpretation of rational numbers. *Elementary Education Online, 7*(3), 693-705.

Yates, G. C. R., & Lyndon, H. L. (2004). Conceptual mediation program in practice: Educational outcomes from two sites. *Research in Science Education, 34*(4), 389-401.

Yin, R. K. (2009). *Case study research: Design and methods.* California: Sage Publications.

Young, J. F. (1985). When practice doesn't make perfect: Improving game performance in secondary level physical education classes. *Journal of Physical Education, Recreation and Dance, 56,* 24-26.

Yusof, J., & Malone, J. (2003). Mathematical errors in fractions: A case of Bruneian primary 5 pupils. In L. Bragg, C. Campbell, G. Herbert, & J. Mousley (Eds.), *Mathematics education research: Innovation, networking, opportunity,* (Proceedings of the 26th Annual Conference of the Mathematics Education Research Group of Australasia, Vol. 2, pp. 650-657). Geelong: Deakin University.

Zazkis, R., & Mamolo, A. (2016). Extending horizon: A story of a teacher educator. In M. B. Wood, E. E. Turner, M. Civil, & J. A. Eli (Eds.), *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education.* Tuscon, AZ: The University of Arizona.

Zeldin, A. L., Britner, S. L., & Pajares, F. (2008). A comparative study of the self-efficacy beliefs of successful men and women in mathematics, science, and technology careers. *Journal of Research in Science Training, 45*(9), 1036-1058.

Appendix A

Fractions Summative Assessments Years 7-9

Fractions Test Year 7 Name ………………………………………

1. Write these as equivalent fractions /6

$$\frac{3}{5} = \frac{\quad}{10} \qquad \frac{16}{20} = \frac{\quad}{5} \qquad \frac{20}{6} = \frac{\quad}{30}$$

$$\frac{44}{100} = \frac{11}{\quad} \qquad \frac{3}{8} = \frac{15}{\quad} = \frac{\quad}{24}$$

2. <u>Simplify</u> these fractions. /5

$$\frac{5}{25} = \qquad \frac{8}{10} = \qquad \frac{12}{30} =$$

$$\frac{25}{40} = \qquad \frac{80}{48} =$$

3. Draw a picture or diagram to explain why $\mathbf{3\frac{2}{3} = \frac{11}{3}}$ /3

4.

Change these <u>mixed numbers</u> to <u>improper fractions</u> /3

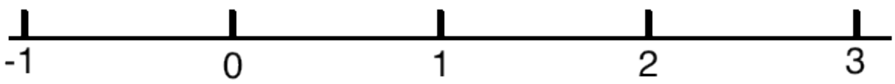
$\mathbf{3\frac{1}{2}} =$ $\mathbf{4\frac{3}{5}} =$ $\mathbf{1\frac{1}{20}} =$

5. Change these <u>improper fractions</u> to <u>mixed numbers</u> /3

$\frac{\mathbf{10}}{\mathbf{3}} =$ $\frac{\mathbf{10}}{\mathbf{2}} =$ $\frac{\mathbf{17}}{\mathbf{4}} =$

6. Clearly mark these fractions on the number line below. /4

$\frac{\mathbf{2}}{\mathbf{4}},$ $\mathbf{2\frac{4}{5}},$ $-\frac{\mathbf{3}}{\mathbf{4}},$ $\frac{\mathbf{5}}{\mathbf{3}}$



7. Put these fractions into order from *smallest to largest*. /4

$$\frac{\mathbf{3}}{\mathbf{10}}, \frac{\mathbf{3}}{\mathbf{5}}, \frac{\mathbf{3}}{\mathbf{4}}, \frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{11}}{\mathbf{20}}$$

8. Use > or < or = in these. e.g 10 > 7 /6

$\frac{5}{6}$ $\frac{4}{5}$ $\frac{3}{4}$ $\frac{3}{5}$ $\frac{12}{15}$ $\frac{4}{5}$

$\frac{12}{18}$ $\frac{4}{6}$ $2\frac{1}{3}$ $1\frac{2}{3}$ $2\frac{4}{5}$ $\frac{24}{5}$

9. A family block of chocolate consists of 12 rows, with each row having 6 squares of chocolate. Paul eats 16 squares.

What fraction of the block, in simplest terms, has Paul eaten? /3

10. Four friends, Mark, John, Kate and Sarah, all competed in an 800m race. Their respective finishing times were: $\boldsymbol{3\frac{1}{3}\ minutes, 3\frac{5}{12} minutes, 3\frac{1}{4} minutes}$ **and** $\boldsymbol{3\frac{4}{15} minutes}$. Use equivalent fractions to compare the finishing times and write down the correct finishing order of the friends.

/4

11. Amy, Barry, Charlotte and Danni order three pizzas. Each pizza is cut into 8 equal slices. Amy eats 3 slices, Barry eats 7 slices and Charlotte and Danni both eat 5 slices.

a. How many slices of pizza were eaten in total? /1

b. How many pizzas were eaten in total? (Give your answer as a mixed number) /1

c. What fraction of pizza, in simplest terms, was left uneaten? /2

12. By making use of equivalent fractions, name a fraction that is half way between one-sixth and one-quarter.

/3

Year 7-Fraction, Decimal and Percentage Test

Name:________________________

1. Write these fractions as decimals. **(4)**

$\frac{3}{10} =$	$\frac{42}{100} =$	$\frac{5}{100} =$	$2\frac{1}{4} =$

2. Write these decimals as fractions in *simplest form*. **(4)**

$0.7 =$	$0.04 =$	$3.25 =$	$0.35 =$

3. Write the following decimal in ascending order (*smallest to largest*). **(3)**

9.9	0.9	0.99	0.909	9

____

4. Complete these calculations. Marks will be awarded to setting out. **(8)**

$0.7 + 1.15$	$29.04 + 0.15 + 0.055$
$3.4 - 1.85$	$4.4 - 0.8$

5. How much change do you get from \$30 if you spend \$3.75, \$2.40 and \$6.95? (Show your calculations)

(4)

6. Complete these calculations. **(6)**

$2.3 \times 10 =$	$0.385 \times 100 =$	$0.9 \times 1000 =$
$2.3 \div 10 =$	$14.5 \div 100 =$	$3 \div 1000 =$

7. Find the total cost of the following purchases. **(4)**
   a. Five Mars Bars were purchased from Coles at a price of $1.80 each.

   b. Two hundred red beads were purchased from The Bead Seller at a cost of forty cents each.

8. Complete the following table. The 1st row is completed as an example. **(8)**

Fraction	Decimal	Percentage
$\frac{1}{2}$	0.5	50%
	0.25	
		20%
$\frac{3}{4}$		
		1%

9. Complete the following fraction calculations.
Show all workings and simplify fractions where possible.

(15)

$\frac{5}{7}+\frac{4}{7}$	$\frac{1}{2}\times\frac{1}{2}$	$\frac{1}{2}\div\frac{1}{4}$
$2\frac{1}{4}+1\frac{3}{8}$	$3\frac{3}{4}-1\frac{1}{4}$	$2\frac{1}{8}-\frac{1}{4}$
$2\frac{1}{2}\times 1\frac{1}{3}$	$8\div 1\frac{1}{3}$	$\frac{9}{10}\times\frac{1}{18}$

10. Which of the following amounts is larger, A or B? (Show all calculations)

(4)

A - $\frac{3}{4}\, of$ \$64 OR **B-** 60% of \$90

11. Charlie has a bag of 80 banana lollies. He then gives 80% of the lollies to Mary. Mary then gives $\frac{3}{8}$ of the lollies she received to Max. Calculate how many lollies Charlie, Mary and Max each have.

Charlie:_________________
Mary:___________________Max:_______________________

YEAR 8 Fractions, Decimals & Percentages TEST

You are not allowed to use a calculator when you answer these questions.

SECTION C (42 marks)

1. **Fill in the gaps, giving the fraction in simplest form:**

Fraction	Decimal	Percentage
	0.75	
$\frac{1}{5}$		
		5%
	0.3	
$\frac{11}{20}$		

(10 Marks)

2. A survey asked coffee drinkers whether they take milk or sugar in their coffee. 16 took milk only, 20 took both milk and sugar, 35 took sugar only and 29 took neither.

   a. How many took part in the survey altogether? ____________

   b. What fraction had milk only? ____________ = ____________ simplest form

   c. What decimal represents milk only? ____________

   d. What percentage had milk only? ____________

(4 Marks)

3. **Give all answers in their simplest form.**

a. $\frac{5}{8} - \frac{1}{8}$	b. $\frac{5}{8} + \frac{1}{8}$	c. $\frac{28}{6}$
d. $\frac{2}{3} + \frac{3}{5}$	e. $\frac{3}{5} \times \frac{2}{3}$	f. $\frac{1}{6} \div \frac{2}{3}$

(6 Marks)

4. **Calculate**

a) 1.732 × 100	**b)** 1.732 ÷ 100	**c)** \$10.00 – \$3.60
d) 1.2 + 9.08	**e)** 1.2 × 6	**f)** 1.46 ÷ 2
g) 4.63 – 2.32	**h)** 0.8 × 0.6	**i)** 2.85 ÷ 0.3

(9 Marks)

5. Fill in the gaps:

$$\frac{2}{6} = \frac{}{3} = \frac{6}{} = \frac{}{30} = \frac{12}{}$$

(4 Marks)

6. Round the following numbers to 2 decimal places

a) 3.14159	1) 2.0984

(2 Marks)

7. Change the following fractions into decimal numbers

a. $\frac{3}{5}$	b. $1\frac{3}{4}$

(2 Marks)

8. Ali went to a market and bought 9.7kg of fresh produce. She bought 3.4kg of watermelon and 2.8kg of grapes. The rest of the produce she bought was bananas.
How many kilos of bananas did Ali buy?

(2 Marks)

9. Michelle scored 38 out of 50 in her test. Kathy scored 14 out of 20 in her test.

a) Convert each of these scores to a percentage

b) To get a "B" on a test a student needs to score between 75% and 85%. Did either of the students earn a "B"?

(3 Marks)

SECTION B/A (35 Marks)

ALL ANSWERS ARE TO SHOW WORKING

1. **a.** Convert each of the following to a decimal

$\frac{7}{20}$ 0.309 $\frac{1}{3}$ 32% $\frac{3}{10}$

b. Arrange the decimal numbers in ascending order (from smallest to largest)

(3 Marks)

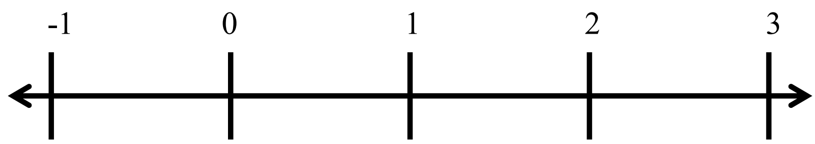
2. **Calculate the following:**

a . $3\frac{2}{3} - 2\frac{3}{4}$	b. $1\frac{3}{4} + 2\frac{1}{6}$	c. $1\frac{3}{4} \times 1\frac{1}{5}$

(3 Marks)

1. Mark these fractions on the number line below.

(a) $\frac{3}{4}$ (b) $1\frac{1}{3}$ (c) $-\frac{1}{4}$ (d) $2\frac{1}{10}$ (e) $\frac{5}{2}$



(5 Marks)

4. Round $0.\dot{6}$ to three decimal places.

(1 Mark)

5. Is the answer to $1.54 \div 6$ a terminating or recurring decimal?

(2 Marks)

6. The ingredients for a Choc-chip biscuit recipe cost \$4.50. To make enough for a group of people, Joe had to make $2\frac{1}{2}$ batches of the recipe.

a. What will be the total cost for the biscuits? (2 Marks)

b. The original recipe included $\frac{3}{4}$ of a cup of flour.

How much flour will be needed for the $2\frac{1}{2}$ batches Joe needs to make?

(2 Marks)

7. Yvonne filled up her car with diesel. The tank took 60 litres at a cost of 165.9 cents per litre. What was the total cost for the diesel?

(3 Marks)

8. It takes Izzy $\frac{3}{4}$ of an hour to lay a row of 50 bricks.

a. How many **rows** will she lay in $4\frac{1}{2}$ hours?
(2 Marks)

b. How many **bricks** will she lay in that time?
(1 Mark)

c. How long will it take her to build a wall made up of 22 **rows**? (2 Marks)

d. How long will it take her to build a section made up of 450 **bricks**? (3 Marks)

9. Change the following to a decimal. Give your answer to 2 decimal places.

$\frac{6}{7}$

(2 Marks)

10. Martin needs to dig a hole $2\frac{7}{8}$ metres deep. When he finished digging it was 2.80m deep. Is it too deep or too shallow, and by how much (in centimetres)?

(4 Marks)

Appendix B

Fractions Diagnostic Test (pre- and post-)

Name:______________________ Class: ______ Year Group:_____

1. For each pair of fractions, either CIRCLE the **LARGER** fraction, OR write = between them

a. $\frac{5}{8}$ $\frac{1}{3}$

b. $\frac{5}{8}$ $\frac{7}{8}$

c. $\frac{2}{3}$ $\frac{3}{4}$

d. $\frac{4}{6}$ $\frac{2}{3}$

e. $\frac{4}{5}$ $\frac{3}{8}$

f. $\frac{4}{7}$ $\frac{4}{9}$

g. $\frac{5}{7}$ $\frac{3}{8}$

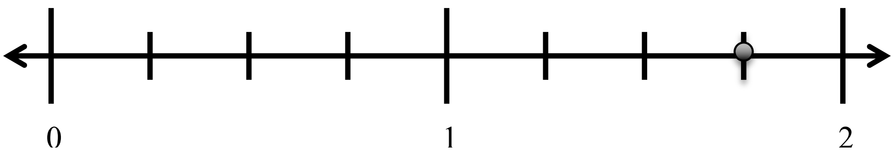
h. $\frac{3}{10}$ $\frac{2}{5}$

i. $\frac{5}{7}$ $\frac{3}{4}$

2. Put these fractions into order from SMALLEST to LARGEST

a. $\frac{3}{10}$, b. $\frac{3}{5}$, c. $\frac{3}{4}$, d. $\frac{1}{2}$, e. $\frac{11}{20}$

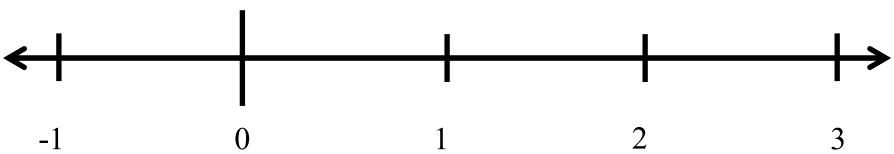
3. Which of the following numbers is the value of the point shown on the number line? (CIRCLE)



a. $\frac{3}{4}$ b. $\frac{7}{8}$ c. $1\frac{7}{8}$ d. $1\frac{3}{4}$

2. Mark these fractions on the number line below.

$a.$ $\frac{3}{4}$ $b.$ $1\frac{1}{3}$ $c.$ $-\frac{1}{4}$ $d.$ $2\frac{1}{10}$ $e.$ $\frac{5}{2}$



Make sure you make it clear which number is which.

3. Calculate:

a. $\frac{2}{3}$ of 9

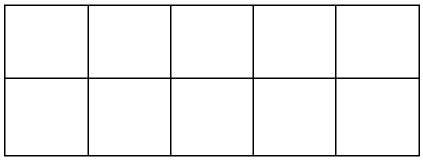
b. $\frac{1}{3}$ of $\frac{1}{2}$

c. If there is $\frac{7}{8}$ of a cake left and 14 people would all like a piece, what fraction of a cake will they receive?

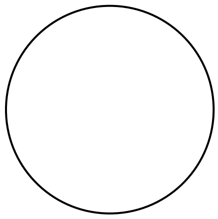
d. How many $\frac{1}{8}$ are there in $3\frac{1}{2}$?

e. Colour in $\frac{3}{4}$ of this shape:

f. Colour in $\frac{2}{3}$ of this shape:



g. Colour in $\frac{2}{5}$ of this this shape:



4. Write these as equivalent fractions

$\frac{3}{5} = \frac{\quad}{10}$, $\frac{12}{18} = \frac{\quad}{6}$, $\frac{1}{3} = \frac{\quad}{18}$

5. Write each fraction in its simplest form

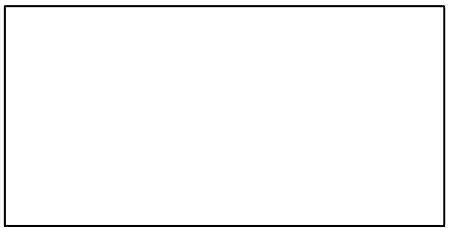
$\frac{3}{6} =$ $\frac{2}{8} =$ $\frac{6}{15} =$ $\frac{8}{1} =$

6. Find the answers to the following problems. Show your working out, where needed because your working out helps you and helps us see what your method is. Give all answers in simplest form.

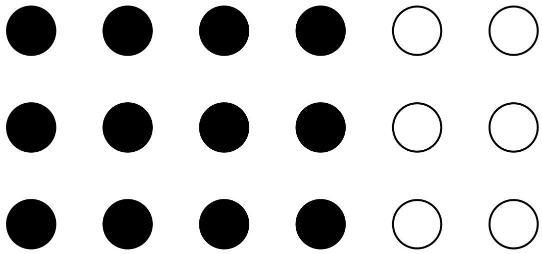
$a.\ \frac{2}{9}+\frac{5}{9}$	$b.\ \frac{1}{2}+\frac{1}{4}$	$c.\ 1\frac{3}{4}+2\frac{1}{6}$
$d.\ \frac{2}{3}+\frac{3}{5}$	$e.\ \frac{3}{4}+\frac{1}{5}$	$f.\ 1\frac{1}{5}+3\frac{4}{5}$
$g.\ \frac{6}{7}-\frac{2}{7}$	$h.\ \frac{9}{16}-\frac{1}{2}$	$i.\ 10-\frac{1}{3}$
$j.\ \frac{1}{2}\times\frac{3}{5}$	$k.\ \frac{2}{3}\times 12$	$l.\ \frac{3}{5}\times\frac{2}{3}$
$m.\ \frac{6}{7}\times\frac{14}{15}$	$n.\ 6\times\frac{1}{3}$	$o.\ 1\frac{3}{4}\times 2\frac{5}{6}$

$p.\ \frac{3}{5} \div \frac{1}{5}$	$q.\ \frac{1}{2} \div \frac{1}{4}$	$r.\ \frac{3}{4} \div \frac{1}{8}$
$s.\ \frac{1}{6} \div \frac{2}{3}$	$t.\ 5 \div \frac{1}{4}$	$u.\ 4 \div \frac{2}{3}$

7. If this is $\frac{2}{3}$ of a shape, draw a shape that shows the whole

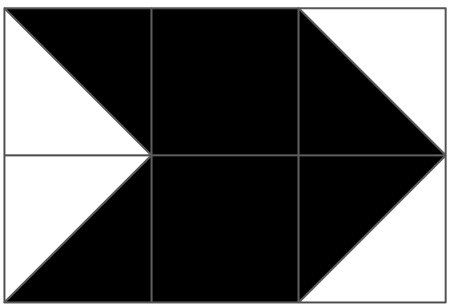


8.

a. What fraction of dots is black?	
	

b. What is another way of writing the same fraction?

9. What fraction of the whole rectangle is shaded?



Appendix C

Fractions Diagnostic Test (Delayed Retention)

Name:____________________________ Class: _______ Year Group:______

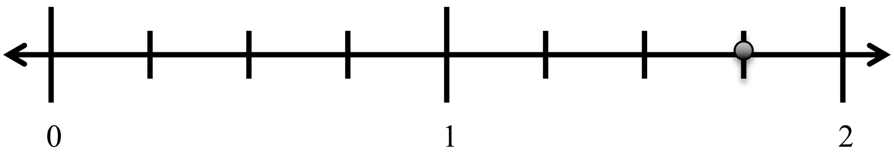
1. For each pair of fractions, either CIRCLE the **LARGER** fraction, OR write = between them. Please also give a brief explanation for each of your answers.

			Reason
a.	$\frac{1}{4}$	$\frac{5}{7}$	
b.	$\frac{8}{9}$	$\frac{5}{9}$	
c.	$\frac{2}{3}$	$\frac{3}{5}$	
d.	$\frac{5}{6}$	$\frac{10}{12}$	
e.	$\frac{3}{7}$	$\frac{5}{9}$	
f.	$\frac{5}{9}$	$\frac{5}{7}$	
g.	$\frac{3}{5}$	$\frac{2}{9}$	
h.	$\frac{2}{3}$	$\frac{5}{9}$	
i.	$\frac{3}{5}$	$\frac{5}{8}$	

2. Put these fractions into order from **SMALLEST to LARGEST**

a. $\frac{2}{3}$, b. $\frac{1}{2}$, c. $\frac{3}{4}$, d. $\frac{5}{6}$, e. $\frac{7}{12}$

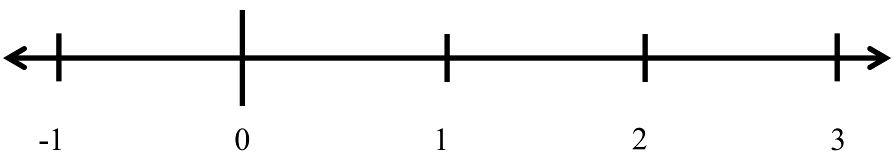
3. Which of the following numbers is the value of the point shown on the number line? (CIRCLE)



a. $\mathbf{\frac{3}{4}}$ b. $\mathbf{\frac{7}{8}}$ c. $1\frac{7}{8}$ d. $1\frac{3}{4}$

4. Mark these fractions on the number line below.

a. $\frac{2}{3}$ *b.* $1\frac{3}{4}$ *c.* $-\frac{3}{4}$ *d.* $2\frac{9}{10}$ *e.* $\frac{7}{3}$



Make sure you make it clear which number is which.

5. Calculate:

a. $\frac{3}{4}$ of 12

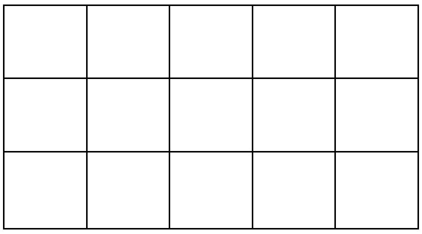
b. $\frac{1}{4}$ of $\frac{1}{3}$

c. If there is $\frac{5}{6}$ of a cake left and 10 people would all like a piece, what fraction of a cake will they receive?

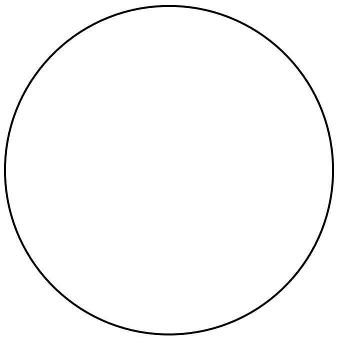
d. How many $\frac{1}{3}$ are there in $4\frac{2}{3}$?

e. Colour in $\frac{3}{5}$ of this shape:

f. Colour in $\frac{3}{4}$ of this shape:



g. Colour in $\frac{2}{7}$ of this shape:



6. Write these as equivalent fractions

$\frac{3}{7} = \frac{\quad}{14}$, $\frac{2}{3} = \frac{\quad}{15}$, $\frac{16}{24} = \frac{\quad}{6}$

7. Write each fraction in its simplest form

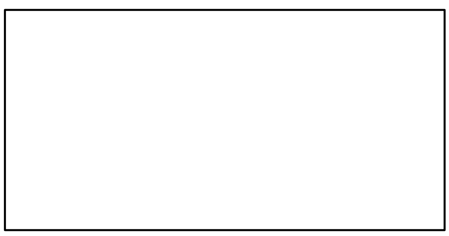
$\frac{6}{12} =$ $\frac{2}{6} =$ $\frac{9}{15} =$ $\frac{6}{1} =$

8. Find the answers to the following problems. Show your working out, where needed because your working out helps you and helps us see what your method is. Give all answers in simplest form.

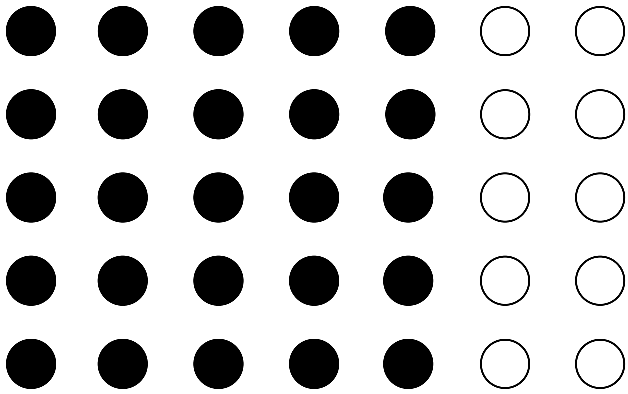
$a.\ \frac{2}{7}+\frac{3}{7}$	$b.\ \frac{1}{3}+\frac{1}{6}$	$c.\ 1\frac{1}{6}+2\frac{3}{4}$
$d.\ \frac{3}{4}+\frac{4}{5}$	$e.\ \frac{2}{3}+\frac{4}{5}$	$f.\ 2\frac{2}{3}+3\frac{1}{3}$
$g.\ \frac{7}{9}-\frac{2}{9}$	$h.\ \frac{8}{9}-\frac{1}{3}$	$i.\ 7-\frac{1}{4}$
$j.\ \frac{1}{4}\times\frac{2}{3}$	$k.\ \frac{3}{4}\times 16$	$l.\ \frac{4}{7}\times\frac{3}{4}$

$m.\ \frac{4}{5} \times \frac{15}{16}$	$n.\ 8 \times \frac{1}{4}$	$o.\ 2\frac{1}{6} \times 1\frac{3}{8}$
$p.\ \frac{4}{7} \div \frac{1}{7}$	$q.\ \frac{1}{2} \div \frac{1}{4}$	$r.\ \frac{2}{3} \div \frac{1}{6}$
$s.\ \frac{1}{8} \div \frac{3}{4}$	$t.\ 4 \div \frac{1}{3}$	$u.\ 6 \div \frac{3}{4}$

9. If this is $\frac{3}{5}$ of a shape, draw a shape that shows the whole

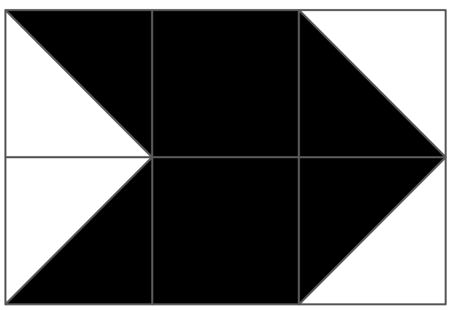


10.

a. What fraction of dots is black?	
	

b. What is another way of writing the same fraction?

11. What fraction of the whole rectangle is shaded?



Appendix D
Self-efficacy Questionnaire

Name: ________________________ Class: _____________ Year Group:_______

This questionnaire is about how you feel about learning maths. Please tick the box that best applies to you for each statement.

	Strongly Disagree	Disagree	Neither or Neutral	Agree	Strongly Agree
1. Working hard leads to success in maths					
2. I look forward to my maths lessons					
3. Some people just cannot do maths					
4. I feel that I can make a start on the problems I have to do in class					
5. There is no point in me trying in maths					
6. I am interested in the things I learn in maths					
7. I cannot change how good I am at maths					
8. When I really try I can get through most difficult tasks					
9. I often get a maths question wrong but I do not understand why					
10. I know if I am going to get a maths question right					
11. I enjoy doing fractions					
12. With fractions, I understand even the most difficult work					
13. I often worry that it will be difficult for me when working with fractions					
14. Even if a fraction problem looks hard, I know I can make progress with it					
15. I get tense when I have to do fractions homework					
16. I like the challenge of a hard fractions problem					
17. I find the teacher's help useful in maths class					
18. When I do better than usual in maths, many times it is because I tried a little bit harder					
19. If I make a mistake in maths, I try to find out where I went wrong					
20. I am the most powerful influence on my own achievement in maths					

Appendix E

Information and Consent Forms

Remediation of Errors with Mathematical Algorithms

Information Sheet for Principal

My name is Alison Manson and I am interested in how best to help students overcome learned errors in fraction computation. I am conducting a PhD project in this area and would like to invite Year 7, 8 and 9 students from your school to participate. This study is being conducted under the supervision of: Associate Professor Helen Chick, Dr Dean Cooley and Associate Professor Karen Swabey, from the Faculty of Education, University of Tasmania.

What is the purpose of this study?
The purpose of this research is to evaluate the effectiveness of the Old Way/New Way (O/N) Technique with systematic error computations in fractions. There is empirical evidence to suggest that O/N is a successful error correction technique for subtraction computation. Despite this, there is a paucity of evidence for O/N's effectiveness with other systematic error computations such as fraction computation. Due to small sample sizes there is a gap in knowledge associated with variables that may cause a response to the O/N. For example, the effect of the O/N technique on correcting mathematical errors is largely unknown for boy and girls. Similarly, it is unknown how the O/N technique affects achievement motivation.

Why has my school been invited to participate?
I am inviting students from your school to participate in this study because I would like access to both male and female students in Years 7, 8 and 9. I need a big group of students so that the study can produce reliable results. Your school's involvement in this study would be entirely on a voluntary basis. There are no consequences if you decide not to participate.

What will my staff and students be asked to do?
All Year 7, 8 and 9 classes will be asked to complete a diagnostic test and a self-efficacy questionnaire and then some students will participate in a remedial mathematics program for a period of approximately four weeks, before completing the diagnostic test again. Students may also be interviewed by the researcher after each remedial lesson. It is anticipated that the mathematics program will take approximately 10-20 minutes out of each timetabled lesson, long tutor or subject support lesson. The focus of the remedial program is to test the effectiveness of the O/N remediation technique compared to conventional methods. It is anticipated that this will empower students with a method of overcoming learned errors or 'bad habits' in computations thus leading to fewer mistakes and, in turn, an increase in motivation. The Year 7, 8 and 9 Maths

teachers will be asked to distribute information sheets and consent forms and collect completed consent forms from students in their class.

Are there any possible benefits from participation in this study?
A focus of the study involves how the O/N technique affects self-efficacy. It is possible that you may notice a positive change in the students undertaking the program as they develop the necessary skills for overcoming learned errors. Students will also benefit from targeted remediation in fractions with the involvement of the researcher delivering targeted interventions. Past research in the area has also reported an increase in engagement associated with studying mathematics. If we are able to take the findings of this small study and link them with wider studies, the result may be valuable information for others and it may inspire other schools to implement similar remediation programs as a way of overcoming habitual errors and increasing motivation.

Are there any possible risks from participation in this study?
There are no specific risks anticipated with participation in this study. The tests and activities are similar to usual classroom activities. Some students may feel mild anxiety as part of doing the diagnostic test, but the test is typical of Year 7, 8 and 9 tests. If you find that any child is becoming distressed by their involvement in the study then it will be recommended that they withdraw. Students will also be encouraged to seek support from their tutor, Head of House and school counsellor, if required.

What if I change my mind during or after the study?
There will be no consequences to you or your school if you decide not to participate. If you decide to discontinue your participation at any time, you may do so without providing an explanation. Any unprocessed data that your students have provided can also be withdrawn if you so desire.

What will happen to the information when this study is over?
All information will be treated in a confidential manner; the school's name, the teachers' names, and the students' names will not be used in any publication arising out of the research. All of the research data will be kept for 5 years in a locked filing cabinet or in password protected computer files, in the Faculty of Education at the University of Tasmania.

How will the results of the study be published?
A summary of the results will be provided to you at the completion of the research. In reporting any findings from this study, no names or places will be mentioned. The student and teacher names will be coded to maintain confidentiality. A summary of results will be made available to participants on Friendsnet.

What if I have questions about this study?
If you would like to discuss any aspect of this study please contact:
Mrs Alison Manson
Email: Alison.Manson@utas.edu.au

Associate Professor Helen Chick
Phone: 62267220
Email: Helen.Chick@utas.edu.au

Dr Dean Cooley
Phone: 63243096
Email: Dean.Cooley@utas.edu.au

Associate Professor Karen Swabey
Phone: 63243512
Email: Karen.Swabey@utas.edu.au

"This study has been approved by the Tasmanian Social Sciences Human Research Ethics Committee. If you have concerns or complaints about the conduct of this study, please contact the Executive Officer of the HREC (Tasmania) Network on (03) 6226 7479 or email human.ethics@utas.edu.au. The Executive Officer is the person nominated to receive complaints from research participants. Please quote ethics reference number [H12766]."
Thank you for taking time to consider this study. You can indicate your consent to being involved in the study by signing the attached consent form. This information sheet is for you to keep.

Remediation of Errors with Mathematical Algorithms

Principal's Consent Form

1. I agree to take part in the research study named above.
2. I have read and understood the Information Sheet for this study.
3. The nature and possible effects of the study have been explained to me.
4. I understand that the study involves Year 7, 8 and 9 students participating in a mathematical diagnostic test, a self-efficacy questionnaire and intervention programs over a four week period for some students.
5. I understand that all research data will be securely stored on the University of Tasmania premises for five years from the publication of the study results, and will then be destroyed.
6. Any questions that I have asked have been answered to my satisfaction.
7. I understand that the researcher(s) will maintain confidentiality and that any information I supply to the researcher(s) will be used only for the purposes of the research.
8. I understand that the results of the study will be published in such a way that I or my school should not be able to be identified as a participant.
9. I understand that my school's participation is voluntary and that I may withdraw the school at any time without any effect.

   If I so wish, I may request that any unprocessed data the school has supplied be withdrawn from the research.

Principal's name: ______________________________

Principal's signature: ______________________________

Date: ____________________

Statement by Investigator

☐ I have explained the project and the implications of participation in it to this volunteer and I believe that the consent is informed and that he/she understands the implications of participation.

If the Investigator has not had an opportunity to talk to participants prior to them participating, the following must be ticked.

☐ The participant has received the Information Sheet where my details have been provided so participants have had the opportunity to contact me prior to consenting to participate in this project.

Investigator's name:

____

Investigator's signature:

____

Date: ________________________

Information Sheet for Parents

My name is Alison Manson and I am an experienced teacher of middle school mathematics. I am interested in how best to help students overcome learned errors in fraction computation and I am conducting a PhD project in this area and would like to invite Year 7, 8 and 9 students from your child's school to participate. This study is being conducted under the supervision of: Associate Professor Helen Chick, Dr Dean Cooley and Associate Professor Karen Swabey, from the Faculty of Education, University of Tasmania.

What is the purpose of this study?
The purpose of this research is to determine if there is an effective way to help students who have difficulties with fraction computation. Specifically, it aims to evaluate the effectiveness of the Old Way/New Way (O/N) Technique, a method by which students are offered a new way of working out fractions and substituting it for their old, incorrect method. The research also aims to determine if using the O/N Technique increases student motivation levels.

Why has my child been invited to participate?
The research focuses specifically on 12-15 year olds and I am inviting all students in Years 7, 8 and 9 from the school. I need a big group of students so that the study can produce reliable results. Only a few students will then be involved in the main part of the study. Your child's involvement in this study would be entirely on a voluntary basis. There are no consequences if you do not want him/her to participate.

What will my child be asked to do?
Part 1: If you allow your son/daughter to participate he/she will be asked to complete a diagnostic fractions test and self-efficacy questionnaire. The test will be like a normal school maths test, conducted during a lesson, and the questionnaire just asks students how they think people learn maths.
Part 2: Depending on the outcome of the test, your child may be invited to participate in a mathematics program for a period of about 4 weeks, intended to help overcome specific fraction errors, and will then complete the diagnostic test again. It is anticipated that the mathematics program will take approximately 10-20 minutes out of each timetabled maths lesson, long tutor or subject support lesson, for 10 lessons. The focus of the remedial program is to test the effectiveness of the O/N remediation technique compared to conventional methods. It is anticipated that this will empower students with a method of overcoming learned errors or 'bad habits' in computations thus leading to fewer mistakes and, in turn, an increase in motivation. The students participating in Part 2 will be briefly interviewed at the end of each lesson, with questions such as, "what did you learn in today's lesson?". There will also be a follow-up diagnostic fractions test later in the year to evaluate long-term retention.

Are there any possible benefits from participation in this study?
A focus of the study involves how the O/N technique affects self-efficacy. It is possible that you may notice a positive change in your child while undertaking the program as they develop the necessary skills for overcoming learned errors.

Are there any possible risks from participation in this study?
There are no specific risks anticipated with participation in this study. However, if you find that your child is becoming distressed they will receive support from their classroom teacher and be reminded that participation is voluntary and may be advised to withdraw. Students will also be encouraged to seek support from their tutor, Head of House and school counsellor, if required.

What if I change my mind during or after the study?
There will be no consequences to your child if they decide not to participate. If you or your child decide to discontinue participation at any time, they may do so without providing an explanation. Any unprocessed data that your child has provided can also be withdrawn if you so desire.

What will happen to the information when this study is over?
Your child will be required to record his/her name and gender on research documents so that they can be identified for remediation if required. However, during the reporting of the study your child will not be identified nor will the school. All of the research data will be kept for 5 years in a locked filing cabinet or in password protected computer files, in the Faculty of Education at the University of Tasmania. After this time the data will be destroyed.

How will the results of the study be published?
The results of the study will be submitted in partial fulfilment of a PhD thesis. A summary of the results will be made available to you at the completion of the research. The summary will be published on the school intranet, FriendsNet. Individual student results will be made available at the request of the parents. In reporting any findings from this study, no names or places will be mentioned. The student and teacher names will be coded to maintain confidentiality.

What if I have questions about this study?
If you would like to discuss any aspect of this study please contact:

Mrs Alison Manson
Email: Alison.Manson@utas.edu.au

Associate Professor Helen Chick
Phone: 62267220
Email: Helen.Chick@utas.edu.au

Dr Dean Cooley
Phone: 63243096
Email: Dean.Cooley@utas.edu.au

Associate Professor Karen Swabey
Phone: 63243512
Email: Karen.Swabey@utas.edu.au

"This study has been approved by the Tasmanian Social Sciences Human Research Ethics Committee. If you have concerns or complaints about the conduct of this study, please contact the Executive Officer of the HREC (Tasmania) Network on (03) 6226 7479 or email human.ethics@utas.edu.au. The Executive Officer is the person nominated to receive complaints from research participants. Please quote ethics reference number [H12766]."

Thank you for taking time to consider this study. You can indicate your consent to your child being involved in the study by signing the attached consent form. This information sheet is for you to keep.

Remediation of Errors with Mathematical Algorithms

Parent's Consent Form

1. I agree for my child to take part in the research study named above.
2. I have read and understood the Information Sheet for this study.
3. The nature and possible effects of the study have been explained to me.
4. I understand that the study involves my child participating in a mathematical diagnostic test, a maths self-efficacy test and may later involve an invitation to participate in intervention programs over a four-week period.
5. I understand that all research data will be securely stored on the University of Tasmania premises for five years from the publication of the study results, and will then be destroyed.
6. Any questions that I have asked have been answered to my satisfaction.
7. I understand that the researcher(s) will maintain confidentiality and that any information my child supplies to the researcher(s) will be used only for the purposes of the research.
8. I understand that the results of the study will be published in such a way that my child cannot be identified as a participant.
9. I understand that my child's participation is voluntary and that he/she may withdraw at any time without any effect.

   If I so wish, I may request that any unprocessed data my child has supplied be withdrawn from the research.

CHILD'S name: ____

Parent's signature: ____

Date: _____________________

Statement by Investigator

☐ I have explained the project and the implications of participation in it to this volunteer and I believe that the consent is informed and that he/she understands the implications of participation.

If the Investigator has not had an opportunity to talk to participants prior to them participating, the following must be ticked.

☐ The participant has received the Information Sheet where my details have been provided so participants have had the opportunity to contact me prior to consenting to participate in this project.

Investigator's name:
____

Investigator's signature:
____

Date: ____________________

Appendix F
Student Work Samples from the O/N Intervention Group and the Traditional Intervention Group

Traditional Intervention Group

Group Makeup
Addition

- 7 students have **addition** errors (add numerator/add denominator, without a finding common denominator)

- 2 students have errors in **incorrect equivalent fractions** in addition

- 1 student with no obvious method for adding fractions, but consistently incorrect:

Multiplication

- 4 students have **multiplication** errors (applied the rule for addition. All converted to common denominator)

- 1 student who adds the numerator and multiplies the denominator

The Plan:

1. Equivalent Fractions – different names for the same thing
2. Same denominators – can only add the same ‘type of thing’ (same base pieces)
3. Different denominators – need to have same type of thing
    - Least common denominator
    - Renaming by equivalent fractions
    - Simplifying
    - Improper fractions/mixed numbers
4. Alternative method for equivalent fractions $\frac{a}{b} + \frac{c}{d} = \frac{(ad + bc)}{bd}$

Addition Questions:

Fraction Basics
EXPLANATION

Common Fractions – are always in reference to a 'whole'
Four Key Concepts:
1. What is the whole or '1'
2. Denominator (bottom) – how many equal sized pieces the whole has been divided into
3. Numerator – how many of those pieces
4. A fraction *is a number* – it has a position on a number line
   e.g. $\frac{3}{4}$ is a whole that was divided into 4 equal parts and we have 3 of those.
   Draw this fraction on a number line in your book (line broken up into four equal parts)

***Equivalent Fractions** have the same value, even though they may look different.*

ACTIVITY 1: Copy into your exercise book the heading: "Equivalent Fractions"

Underneath, write, "Different names for the same thing"

These fractions are really the same:

$$\frac{1}{2} = \frac{2}{4} = \frac{4}{8}$$

EXPLANATION:

Why are they the same?

USING MANIPULATIVES – FRACTION STRIPS AND FRACTION WALLS

Cut out each strip, lengthways.

*Fold each strip along the vertical lines. Notice how the same 'whole' can be divided into different sized pieces – this is the **denominator***

Label each part of the individual strip with the size of the portion. Each piece must be labeled as one part of that piece e.g. $\frac{1}{2}$ and $\frac{1}{2}$ (<u>NOT</u> $\frac{1}{2}$ *and* $\frac{2}{2}$)

The number of these portions is the **numerator.**

On a blank fraction wall – name your fraction wall by labelling each piece. Now you are going to compare different fractions. Colour in fractions that are the same as $\frac{1}{2}$ i.e. $(\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10})$ use the same coloured pencil.

With a different coloured pencil, colour in fractions that are the same as $\frac{1}{3}$

REVIEW

Equivalent Fractions *have the same value, even though they may look different.*

<u>Because</u> when you multiply or divide **both** the top and bottom by the same number, the fraction keeps it's value.

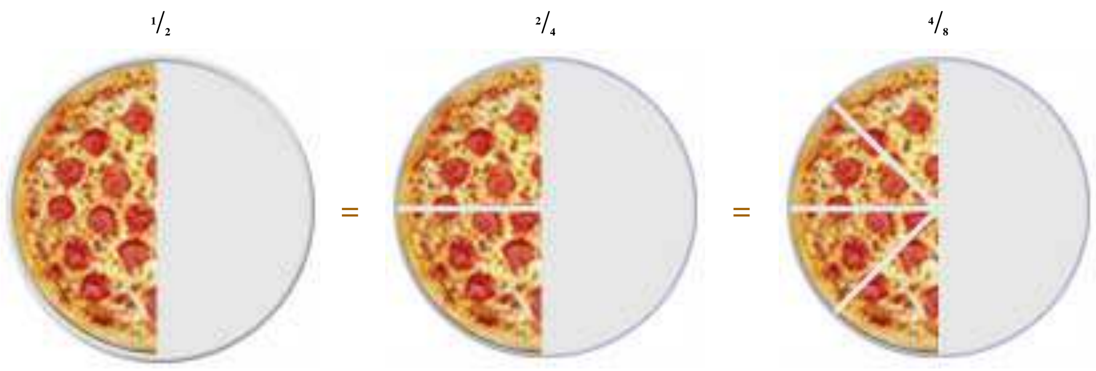
The rule to remember is:

"Change the bottom using multiply or divide,
And the same to the top must be applied"

So, here is why those fractions are really the same:

$$\frac{1}{2} \overset{\times 2}{\underset{\times 2}{=}} \frac{2}{4} \overset{\times 2}{\underset{\times 2}{=}} \frac{4}{8}$$

And visually it looks like this:



Dividing

Here are some more equivalent fractions, this time by dividing:

÷ 3 ÷ 6

$$\frac{18}{36} = \frac{6}{12} = \frac{1}{2}$$

÷ 3 ÷ 6

Choose the number you divide by carefully, so that the results (both top and bottom) stay whole numbers.

If we keep dividing until we can't go any further, then we have simplified the fraction (made it as simple as possible).

PLAY: THE FRACTIONS BOARD GAME

MULTIPLICATION

ESTABLISH

1. Establish what students understand – get them to write in their workbook
   - What does it mean to multiply two fractions together
   - how, in words, they multiply two fractions together.

DISCUSS - Today we are going to look at these 'true' statements:

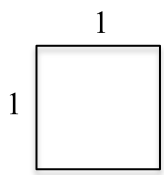
1. Multiplication is the same as repeated addition when you add the same number again and again.

2. Times means "groups of."

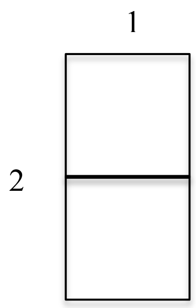
3. A multiplication problem can be shown as a rectangle.

TRY – 'It's in the Fold' activity

- *Multiply using rectangles –*
  - *Draw rectangle on the whiteboard, label each side with '1'.*



- *Show 1 x 1 = 1*
- *Show 2 x 1 = 2*



- *Now show* $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$